

Modular elliptic directions with Complex Multiplication (with an application to Gross's elliptic curves)

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October 23, 2008

Abstract

Let A_f be the abelian variety attached by Shimura to a normalized newform $f \in S_2(\Gamma_1(N))$ and assume that A_f has elliptic quotients. The paper deals with the determination of the one dimensional subspaces (elliptic directions) in $S_2(\Gamma_1(N))$ corresponding to the pullbacks of the regular differentials of all elliptic quotients of A_f . For modular elliptic curves over number fields without complex multiplication (CM), the directions were studied by the authors in [8]. The main goal of the present paper is to characterize the directions corresponding to elliptic curves with CM. Then, we apply the results obtained to the case $N = p^2$, for primes $p > 3$ and $p \equiv 3 \pmod{4}$. For this case we prove that if f has CM, then all optimal elliptic quotients of A_f are also optimal in the sense that its endomorphism ring is the maximal order of $\mathbb{Q}(\sqrt{-p})$. Moreover, if f has trivial Nebentypus then all optimal quotients are Gross's elliptic curve $A(p)$ and its Galois conjugates. Among all modular parametrizations $J_0(p^2) \rightarrow A(p)$, we describe a canonical one and discuss some of its properties.

1 Introduction

Let \mathbb{Q}^{alg} be a fixed algebraic closure of \mathbb{Q} . An elliptic curve C defined over \mathbb{Q}^{alg} is said to be modular if there is a non-constant homomorphism $\pi: J_1(N) \rightarrow C$, where $J_1(N)$ denotes the jacobian of the modular curve $X_1(N)$. Every modular elliptic curve over \mathbb{Q}^{alg} is a quotient of some modular abelian variety A_f attached by Shimura to a normalized newform f . From now on, we shall always consider parametrizations $\pi: J_1(N) \rightarrow C$ which factorize through such abelian varieties A_f , called in this paper modular abelian varieties of *elliptic type*.

A modular parametrization $\pi: J_1(N) \rightarrow C$ defined over a number field $L \subseteq \mathbb{Q}^{\text{alg}}$ induces an injection $\pi^*: \Omega^1(C/L) \hookrightarrow \Omega^1(J_1(N)/L)$. In what follows, we shall identify $\Omega^1(J_1(N)/L)$ with the subspace of cusp forms in $S_2(\Gamma_1(N))$ whose q -expansion lies in $L[[q]]$, via $hdq/q \mapsto h$ where $q = \exp(2\pi iz)$.

The determination of the normalized cusp forms in $S_2(\Gamma_1(N))$ associated with the pullbacks $\pi^*(\Omega^1(C))$ was discussed by the authors in [8] for elliptic curves without complex multiplication. In this paper, we shall deal with the complex multiplication case that needs techniques *ad hoc*. The present case is substantially richer since it requires the intervention of class field theory as well as the main theorem of complex multiplication.

Shimura shows in [16] that all elliptic curves with complex multiplication (CM) are modular. Due to Ribet [12], we know that A_f has an elliptic quotient with CM by an imaginary quadratic field $K \subset \mathbb{Q}^{\text{alg}}$ if and only if $f = f \otimes \chi$, where χ is the quadratic Dirichlet character attached to K . In this case, there is a primitive Hecke character $\psi: I(\mathfrak{m}) \rightarrow \mathbb{Q}^{\text{alg}}$ of conductor an ideal \mathfrak{m} of K such that the q -expansion of the CM normalized newform f is given by

$$f = \sum_{(\mathfrak{a}, \mathfrak{m})=1} \psi(\mathfrak{a}) q^{N(\mathfrak{a})} = \sum_{n=1}^{\infty} a_n q^n.$$

Here, $I(\mathfrak{m})$ denotes the multiplicative group of fractional ideals of K relatively prime to \mathfrak{m} , and the first summation is over integral ideals. The level of f is $N = N(\mathfrak{m}) |\Delta_K|$, the norm of \mathfrak{m} times the absolute value of the discriminant of K . We consider the number fields $E_f = \mathbb{Q}(\{a_n\})$ and $E = \mathbb{Q}(\{\psi(\mathfrak{a})\})$, generated by the images of ψ . One has $E = E_f \cdot K$, and we shall denote by Φ the set of its K -embeddings $E \hookrightarrow \mathbb{Q}^{\text{alg}}$. The number field E is a CM field. Through the paper, for all CM fields we shall denote by $\bar{}$ the canonical complex conjugation.

For future use, we recall that an abelian variety Y is called optimal quotient of an abelian variety X over a field k if there is a surjective morphism $\pi: X \rightarrow Y$ defined over k whose kernel is an abelian variety. In this case, every endomorphism of X which leaves stable $\ker \pi$ induces an endomorphism of Y . The property of being optimal quotient is transitive. Hereafter, every A_f is taken to be optimal quotient of $J_1(N)$.

The plan of the paper is as follows. In section 2, we study the decomposition of A_f over the quadratic field K for f with CM as before. This is an intermediate step necessary to determine the elliptic directions we are interested in. We shall prove

Theorem 1.1 *Let $f \in S_2(\Gamma_1(N))$ be a newform with CM and keep the above notations. There is an abelian variety (A, ι) of CM type Φ defined over K , with $\iota: E \hookrightarrow \text{End}_K^0(A)$, satisfying the following properties:*

- (i) *A is an optimal quotient of A_f over K and the pullback of $\Omega^1(A)$ corresponds with the subspace generated by $\{\sigma f: \sigma \in \Phi\}$;*
- (ii) *$\iota(\psi(\mathfrak{a}))^*(\sigma f) = \sigma \psi(\mathfrak{a}) \sigma f$, for all $\mathfrak{a} \in I(\mathfrak{m})$ and $\sigma \in \Phi$;*
- (iii) *ι is an isomorphism;*
- (iv) *if \mathfrak{p} is a prime ideal of K with $\mathfrak{p} \nmid N$, then the lifting of the Frobenius endomorphism acting on the reduction of $A \bmod \mathfrak{p}$ is $\iota(\psi(\mathfrak{p}))$ or $\iota(\overline{\psi(\mathfrak{p})})$ depending on $K \not\subseteq E_f$ or $K \subseteq E_f$, respectively.*

We remark that the above abelian variety A is simple over K , and that A is A_f over K when $K \not\subseteq E_f$, while A_f is isogenous over K to $A \times \overline{A}$ when $K \subseteq E_f$. To encode both cases of part (iv) in Theorem 1.1, we shall denote by ψ' the primitive Hecke character mod $\overline{\mathfrak{m}}$ defined as

$$\psi'(\mathfrak{a}) = \begin{cases} \psi(\mathfrak{a}), & \text{if } K \not\subseteq E_f; \\ \overline{\psi(\overline{\mathfrak{a}})}, & \text{if } K \subseteq E_f. \end{cases}$$

As it will be shown, one has $\mathfrak{m} = \overline{\mathfrak{m}}$ in the first case.

Then we study the splitting field of A ; that is, the smallest number field where all endomorphisms of A are defined. We make use of class field theory to build a certain abelian extension L/K attached to the Hecke character ψ' ; the field L is a cyclic extension of the Hilbert class field of K and it is contained in the ray class field mod $\overline{\mathfrak{m}}$. To simplify notation, the Artin automorphism $(\frac{L/K}{\mathfrak{a}})$ in $\text{Gal}(L/K)$ will be often denoted by the same symbol representing the ideal \mathfrak{a} . In particular, one has

$$\mathfrak{p}\beta \equiv \beta^{N(\mathfrak{p})} \pmod{\mathfrak{P}}$$

for all $\beta \in \mathcal{O}_L$, where \mathfrak{P} is an unramified prime ideal of L over a prime ideal \mathfrak{p} of K . The extension L/K is characterized by the property that \mathfrak{a} viewed in $\text{Gal}(L/K)$ is trivial if and only if $\psi'(\mathfrak{a}) \in K^*$. The main result of Section 3 is the following

Theorem 1.2 *Let A be as above. Then,*

- (i) *there is an elliptic curve C defined over L with complex multiplication by the ring of integers \mathcal{O}_K and such that A is isogenous over L to $C^{\dim A}$;*
- (ii) *the field L is the smallest number field satisfying $\text{End}_{\mathbb{Q}^{\text{alg}}}^0(A) = \text{End}_L^0(A)$;*
- (iii) *there is a one-cocycle $\lambda: I(\overline{\mathfrak{m}}) \rightarrow L^*$ satisfying $\lambda(\mathfrak{a}) = \psi'(\mathfrak{a})$ for all $\mathfrak{a} \in I(\overline{\mathfrak{m}})$ with $(\frac{L/K}{\mathfrak{a}}) = \text{id}$ in $\text{Gal}(L/K)$. The class of λ in $H^1(I(\overline{\mathfrak{m}}), L^*)$ is uniquely determined by this condition.*

In view of (iii), the cohomology class of λ depends intrinsically on A , and we shall denote it by $[A] \in H^1(I(\overline{\mathfrak{m}}), L^*)$. Section 4 is devoted to determining the elliptic directions in $\Omega^1(A)$ in terms of $[A]$. To this end, for each one-cocycle $\lambda \in [A]$ and $\sigma \in \Phi$, we introduce the sums

$$g_\sigma(\lambda) := \sum_{\mathfrak{a} \in \text{Gal}(L/K)} \frac{\mathfrak{a}^{-1} \lambda(\mathfrak{a})}{\sigma \psi'(\mathfrak{a})} \in {}^\sigma E \cdot L,$$

and also its Φ -trace

$$\text{tr}_\Phi(\lambda) := \sum_{\sigma \in \Phi} g_\sigma(\lambda) \in L.$$

Theorem 1.3 *With the above notations, we have:*

- (1) *if $\sum_{n \geq 1} \gamma_n q^n \in S_2(\Gamma_1(N))$ corresponds to an elliptic direction attached to a modular parametrization $\pi \in \text{Hom}_L(A, C)$, then $\gamma_1 \neq 0$.*
- (2) *The following statements are equivalent:*

- (i) *the normalized cusp form*

$$h = q + \sum_{n \geq 2} \gamma_n q^n \in S_2(\Gamma_1(N))$$

gives an elliptic direction attached to some $\pi \in \text{Hom}_L(A, C)$;

- (ii) *there is a one-cocycle $\lambda \in [A]$ with $\text{tr}_\Phi(\lambda) = [L : K]$ and such that*

$$h = \frac{1}{[L : K]} \sum_{\sigma \in \Phi} g_\sigma(\lambda) \cdot {}^\sigma f.$$

The q -expansion of this elliptic direction is then given by

$$h = \begin{cases} \sum_{(\mathfrak{a}, \mathfrak{m})=1} \mathfrak{a}^{-1} \lambda(\mathfrak{a}) q^{N(\mathfrak{a})} & , \text{ if } K \not\subseteq E_f; \\ \sum_{(\mathfrak{a}, \mathfrak{m})=1} \frac{N(\mathfrak{a})}{\lambda(\overline{\mathfrak{a}})} q^{N(\mathfrak{a})} & , \text{ if } K \subseteq E_f. \end{cases}$$

Moreover, all other elliptic directions are $\iota(\mathfrak{a})^(h)$, for $\mathfrak{a} \in E^*$, and it holds the equality $\iota(\psi'(\mathfrak{a}))^* h = \mathfrak{a}^{-1} \lambda(\mathfrak{a}) \mathfrak{a}^{-1} h$ for every $\mathfrak{a} \in I(\overline{\mathfrak{m}})$.*

We shall say that a one-cocycle $\lambda \in [A]$ is *modular* if one has $\text{tr}_\Phi(\lambda) = [L : K]$. According to Theorem 1.3, these are precisely the one-cocycles that provide the elliptic directions. In Section 3, we also describe how to obtain all modular one-cocycles in $[A]$ explicitly way by means of a K -linear projector, and close the section by raising some open questions.

In the last three sections, we deal with the particular case concerning the level $N = p^2$ where $p > 3$ is a prime with $p \equiv 3 \pmod{4}$. The relevance of this case is in connection with the elliptic curves $A(p)$ studied by Gross in [9] and [10]. For convenience of the reader, we recall here its definition. Let $K = \mathbb{Q}(\sqrt{-p})$ and let \mathcal{O}_K be its ring of integers. Let H denote the Hilbert class field of K , and let $H_0 = \mathbb{Q}(j(\mathcal{O}_K))$ be its maximal real subfield. The elliptic curve $A(p)$ is defined over H_0 and given by the Weierstrass equation

$$y^2 = x^3 + \frac{mp}{2^4 \cdot 3} x - \frac{np^2}{2^5 \cdot 3^3},$$

where m and n are the real numbers satisfying

$$m^3 = j(\mathcal{O}_K), \quad n^2 = \frac{j(\mathcal{O}_K) - 1728}{-p}, \quad \text{sgn } n = \left(\frac{2}{p}\right).$$

The elliptic curve $A(p)$ admits a global minimal model over H_0 with discriminant $-p^3$ and whose invariants are $c_4 = -mp$ and $c_6 = np^2$.

Given any intermediate modular subgroup Γ between $\Gamma_1(p^2)$ and $\Gamma_0(p^2)$ and a normalized newform $f \in S_2(\Gamma)$, we denote by $A_f^{(\Gamma)}$ its associated optimal quotient of $\text{Jac}(X_\Gamma)$, where X_Γ denotes the modular curve over \mathbb{Q} attached to Γ . According to this terminology, we have $A_f^{(\Gamma_1(p^2))} = A_f$. In Section 5, we prove:

Theorem 1.4 *With the above notations, we have:*

- (i) *for every positive divisor d of $(p-1)/2$ there is a unique abelian variety A_f of CM elliptic type in $J_1(p^2)$ such that the Nebentypus of f has order d ; one has $K \not\subseteq E_f$, $\dim A_f = [H : K]\varphi(d)$, where φ is the Euler function, and the splitting field of A_f is the intermediate field between H and $H \cdot \mathbb{Q}(e^{2\pi i/p})$ of degree d .*
- (ii) *Let f be a CM normalized newform in $S_2(\Gamma_1(p^2))$ and let Γ satisfying*

$$\Gamma_1(p^2) \subseteq \Gamma \subseteq \Gamma_\varepsilon := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p^2) : \varepsilon(d) = 1 \right\},$$

where ε is the Nebentypus of f . Then, all optimal elliptic quotients of $A_f^{(\Gamma)}$ have complex multiplication by \mathcal{O}_K . Moreover, if f belongs to $S_2(\Gamma_0(p^2))$, then all optimal quotients of $A_f^{(\Gamma)}$ are defined over H and are precisely the elliptic curve $A(p)$ and its Galois conjugates.

Among all modular parametrizations $J_0(p^2) \rightarrow A(p)$ one stands out. In Section 6, we discuss this canonical parametrization and give some of its arithmetical properties.

Theorem 1.5 *Set $\mathfrak{p} = \sqrt{-p} \mathcal{O}_K$. Let $\delta : I(\mathfrak{p}) \rightarrow H$ be the unique map defined by the conditions $\delta(\mathfrak{a})^{12} = \Delta(\mathcal{O})/\Delta(\mathfrak{a})$ and $\left(\frac{N_{H/K}(\delta(\mathfrak{a}))}{\mathfrak{p}}\right) = 1$. Let ω denote a Néron differential of $A(p)$. Then,*

- (i) *there is an optimal quotient $\pi : J_0(p^2) \rightarrow A(p)$ such that $\pi^*(\omega) = c g(q) dq/q$ where the elliptic direction is given by*

$$g(q) = \sum_{(\mathfrak{a}, \mathfrak{p})=1} \delta(\mathfrak{a}) q^{N(\mathfrak{a})} \in S_2(\Gamma_0(p^2)),$$

and $c \in \mathbb{Z}$ is a unit in $\mathbb{Z}[\frac{1}{2p}]$.

(ii) The complex lattice $\left\{ 2\pi i \int_{\gamma} g(z) dz : \gamma \in H_1(X_0(p^2), \mathbb{Z}) \right\}$ is

$$\frac{1}{c} \cdot i^{(p+1)/4} \cdot \sqrt[h]{\rho \cdot (2\pi)^{(2h+1-p)/4} \cdot \sqrt{p}^{(1-3h)/2} \cdot \prod_{\substack{1 \leq m < p \\ \chi(m)=1}} \Gamma\left(\frac{m}{p}\right) \cdot \mathcal{O}_K}$$

where h is the class number of K , the h -th root is taken to be real, Γ is the Gamma function, and $\rho = \prod_{\mathfrak{a} \in \text{Gal}(H/K)} \frac{\delta(\mathfrak{a})}{\sqrt{N(\mathfrak{a})}}$ is a positive unit of H_0 .

Finally, in Section 7 we discuss how to compute the modular elliptic directions for A_f when $f \in S_2(\Gamma_1(p^2))$ has CM and its Nebentypus is nontrivial.

2 The abelian variety A

We shall adhere to the notations in the Introduction and prove Theorem 1.1. Let $\psi: I(\mathfrak{m}) \rightarrow \mathbb{Q}^{\text{alg}}$ be the fixed primitive Hecke character, and let

$$f = \sum_{(\mathfrak{a}, \mathfrak{m})=1} \psi(\mathfrak{a}) q^{N(\mathfrak{a})} = \sum_{n=1}^{\infty} a_n q^n$$

be its associated CM newform in $S_2(\Gamma_1(N))$. The optimal quotient A_f of $J_1(N)$ is defined over \mathbb{Q} by $A_f = J_1(N)/I_f(J_1(N))$, where $I_f(J_1(N))$ is the annihilator of f in the Hecke algebra acting on $J_1(N)$. In particular, the pullback of $\Omega^1(A_f/\mathbb{Q}^{\text{alg}})$ is $\langle \{\sigma f\} \rangle$ where σ runs over $\text{Gal}(\mathbb{Q}^{\text{alg}}/\mathbb{Q})$. Recall that $E_f = \mathbb{Q}(\{a_n\})$ and $E = \mathbb{Q}(\{\psi(\mathfrak{a})\})$. We fix an isomorphism

$$\iota: E_f \hookrightarrow \text{End}_{\mathbb{Q}}^0(A_f),$$

in such a way that $\iota(a_n)$ corresponds to the Hecke operator T_n acting on A_f . The Nebentypus of f is the mod N Dirichlet character $\varepsilon(d) = \chi(d)\psi((d))/d$, where χ is the quadratic character attached to K . We recall that $\iota(\varepsilon(d))$ is the diamond operator $\langle d \rangle$ acting on A_f . One has

$$\dim A_f = [E_f : \mathbb{Q}] = \begin{cases} [E : K], & \text{if } K \not\subseteq E_f; \\ 2[E : K], & \text{if } K \subseteq E_f. \end{cases}$$

Notice that $E = K \cdot E_f$. Now, we proceed to construct the abelian variety A over K of dimension $[E : K]$ with the properties required in Theorem 1.1. According to Shimura's Proposition 8 in [17], there exists $u \in \text{End}_K^0(A_f)$ such that

$$u^*(\sigma f) = \sqrt{\Delta_K} \cdot \sigma f$$

for all σ in $\text{Gal}(\mathbb{Q}^{\text{alg}}/\mathbb{Q})$. Here, the choice of the square root $\sqrt{\Delta_K}$ fixes u up to a sign. For the case $K \not\subseteq E_f$, we let $A = A_f$ and extend ι to E ,

$$\iota: E \hookrightarrow \text{End}_K^0(A_f),$$

via $\iota(\sqrt{\Delta_K}) = u$. For the second case, we proceed as follows. Since now $K \subseteq E_f$, there is $\alpha \in E_f$ such that $\iota(\alpha) \in \text{End}_{\mathbb{Q}}^0(A_f)$ acts as

$$\iota(\alpha)^*(\sigma f) = \sigma \sqrt{\Delta_K} \cdot \sigma f$$

for all σ in $\text{Gal}(\mathbb{Q}^{\text{alg}}/\mathbb{Q})$. Then, consider the involution $w := \iota(\alpha)u^{-1} \in \text{End}_K^0(A_f)$. Let A be the optimal quotient of $J_1(N)$ defined by A_f/B , where $B = (1-w)A_f$. Clearly, the abelian variety A is defined over K , and $\Omega^1(A/K)$ is identified with $\langle \sigma f \rangle_{\sigma \in \Phi}$. Since B is stable by $\iota(E)$, the isomorphism $\iota: E \hookrightarrow \text{End}_{\mathbb{Q}}^0(A_f)$ induces in a natural way an embedding still denoted by the same letter

$$\iota: E \hookrightarrow \text{End}_K^0(A)$$

such that $\iota(\gamma)^*(\sigma f) = \sigma \gamma \cdot \sigma f$ for all γ in E and all K -embeddings σ in Φ . From the equality $\bar{w} = -w$, it follows that $\bar{B} = (1+w)A_f$. Note that \bar{B} is K -isogenous to A .

A case-by-case argument, employing that $\text{End}_K^0(X) \hookrightarrow \text{End}_{\mathbb{Q}}^0(\text{Res}_{K/\mathbb{Q}}(X))$ for any abelian variety X/K , shows that the abelian variety A is K -simple in both cases. Therefore, it follows that ι is an isomorphism. In both cases, A is an abelian variety of CM type Φ and satisfies (i), (ii), and (iii) of Theorem 1.1.

To conclude the proof, it remains to check the property (iv) relative to the Frobenius liftings. To this end, let p be a prime such that $p \nmid N$ and denote by Frob_p and Ver_p the Frobenius and the Verschiebung acting on the reduction of A_f modulo p , which satisfy $\text{Frob}_p \cdot \text{Ver}_p = p$. By the Eichler-Shimura congruence, we know that

$$\widetilde{T}_p = \text{Frob}_p + \text{Ver}_p \cdot \langle p \rangle,$$

where \widetilde{T}_p and $\langle p \rangle$ denote the reductions of the Hecke operator T_p and the diamond operator $\langle p \rangle$ acting on $A_f \bmod p$. Let us consider the two cases separately.

Case $K \not\subseteq E_f$: first, assume that $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K . Since

$$\iota(a_p) = \iota(\psi(\mathfrak{p})) + \iota(\psi(\bar{\mathfrak{p}})), \quad \iota(\psi(\mathfrak{p})) \cdot \iota(\psi(\bar{\mathfrak{p}})) = p \langle p \rangle,$$

and $\widetilde{T}_p = \widetilde{\iota(a_p)}$, it follows that the lifting of Frob_p is either $\iota(\psi(\mathfrak{p}))$ or $\iota(\psi(\bar{\mathfrak{p}}))$. Since a certain power of $\psi(\mathfrak{p})$ belongs to \mathfrak{p} , one concludes that the lifting of $\text{Frob}_{\mathfrak{p}} = \text{Frob}_p$ is $\iota(\psi(\mathfrak{p}))$. A similar argument works when $p\mathcal{O}_K = \mathfrak{p}$ is inert in K , taking into account that $\text{Frob}_{\mathfrak{p}} = \text{Frob}_p^2 = -p \langle p \rangle = \iota(\psi(\bar{\mathfrak{p}}))$.

Case $K \subseteq E_f$: since $\iota(E)$ leaves the abelian subvariety B stable, applying the same arguments as before, it follows that $\iota(\psi(\mathfrak{p}))$ is the lifting of $\text{Frob}_{\mathfrak{p}}$ acting on the reduction of $B \bmod \mathfrak{p}$. Since A is K -isogenous to \bar{B} , the statement (iv) holds in this case as well. This completes the proof of Theorem 1.1.

The following lemma will be used in the next sections.

Lemma 2.1 *If $K \not\subseteq E_f$, then $\mathfrak{m} = \overline{\mathfrak{m}}$.*

Proof. Since $K \not\subseteq E_f$, there is σ in $\text{Gal}(\mathbb{Q}^{\text{alg}}/K)$ such that $\sigma f = \overline{f}$. First, we prove that the Hecke characters ${}^\sigma\psi$ and ψ_c given by ${}^\sigma\psi(\mathfrak{a}) = \sigma(\psi(\mathfrak{a}))$ and $\psi_c(\mathfrak{a}) = \overline{\psi(\overline{\mathfrak{a}})}$ coincide on $I(\mathfrak{m}\overline{\mathfrak{m}})$. Indeed, since ${}^\sigma\varepsilon = \varepsilon^{-1}$ the assertion is immediate for prime ideals $\mathfrak{p} \mid p$ when p is inert. For the case that p splits completely in K , from the equalities ${}^\sigma a_p = \overline{a_p}$ and ${}^\sigma\varepsilon(p) = \varepsilon^{-1}(p)$, that is,

$${}^\sigma\psi(\mathfrak{p}) + {}^\sigma\psi(\overline{\mathfrak{p}}) = \psi_c(\mathfrak{p}) + \psi_c(\overline{\mathfrak{p}}) \quad \text{and} \quad {}^\sigma\psi(\mathfrak{p}) \cdot {}^\sigma\psi(\overline{\mathfrak{p}}) = \psi_c(\mathfrak{p}) \cdot \psi_c(\overline{\mathfrak{p}}),$$

it follows that ${}^\sigma\psi(\mathfrak{p})$ is either $\psi_c(\mathfrak{p})$ or $\psi_c(\overline{\mathfrak{p}})$. Again, we obtain that ${}^\sigma\psi(\mathfrak{p})$ and $\psi_c(\mathfrak{p})$ are equal because a certain power of them lie in \mathfrak{p} . Both Hecke characters being primitive of conductor \mathfrak{m} and $\overline{\mathfrak{m}}$ respectively, we must have $\mathfrak{m} = \overline{\mathfrak{m}}$. \square

3 Splitting field of A

We first introduce an abelian extension L/K that will play a key role in the splitting of the abelian variety A over \mathbb{Q}^{alg} . Let ψ' be the primitive Hecke character mod $\overline{\mathfrak{m}}$,

$$\psi': I(\overline{\mathfrak{m}}) \rightarrow \mathbb{Q}^{\text{alg}},$$

given by $\psi'(\mathfrak{a}) = \psi(\mathfrak{a})$ if $K \not\subseteq E_f$ or $\psi'(\mathfrak{a}) = \overline{\psi(\overline{\mathfrak{a}})}$ otherwise. We consider the character $\eta: (\mathcal{O}_K/\overline{\mathfrak{m}})^* \rightarrow \mathbb{Q}^{\text{alg}}$ defined by

$$\eta(a) = \frac{\psi'((a))}{a}, \quad \text{for all } a \in \mathcal{O}_K \text{ with } (a, \overline{\mathfrak{m}}) = 1.$$

One easily checks that η is well-defined. Recall that the existence of a Hecke character mod $\overline{\mathfrak{m}}$ is equivalent to the condition that the composition $\mathcal{O}_K^* \hookrightarrow \mathcal{O}_K \rightarrow \mathcal{O}_K/\overline{\mathfrak{m}}$ is a group monomorphism (see [16]) and thus $\ker \eta \cap \mathcal{O}_K^* = \{1\}$. By class field theory, to the congruence subgroup

$$P_\eta(\overline{\mathfrak{m}}) = \{(a) \in I(\overline{\mathfrak{m}}): a \bmod \overline{\mathfrak{m}} \in \ker(\eta)\}$$

it corresponds an abelian extension L/K . It is easy to check that, for $\mathfrak{a} \in I(\overline{\mathfrak{m}})$, one has $\mathfrak{a} \in P_\eta(\overline{\mathfrak{m}})$ if and only if $\psi'(\mathfrak{a}) \in K$. Let $K_{\overline{\mathfrak{m}}}$ denote the ray class field of K mod $\overline{\mathfrak{m}}$. Since the map $a \mapsto a\mathcal{O}_K$ provides an isomorphism between $\ker \eta$ and $P_\eta(\overline{\mathfrak{m}})/P_1(\overline{\mathfrak{m}})$, by using the exact sequence

$$1 \rightarrow \mathcal{O}_K^* \rightarrow (\mathcal{O}_K/\overline{\mathfrak{m}})^* \rightarrow I(\overline{\mathfrak{m}})/P_1(\overline{\mathfrak{m}}) \rightarrow I(\mathcal{O}_K)/P(\mathcal{O}_K) \rightarrow 1,$$

one readily shows that $L = K_{\overline{\mathfrak{m}}}^{\ker \eta}$ and $\text{Gal}(L/H)$ is isomorphic to the cyclic group $\text{im}(\eta)/\mathcal{O}_K^*$. Recall that here H denotes the Hilbert class field of K and, as usual, for any integral ideal \mathfrak{n} we denote by $P(\mathfrak{n})$ the subgroup of $I(\mathfrak{n})$ formed by principal ideals and the subscript 1 is for the subgroup of principal ideals with a generator congruent

to one mod \mathfrak{n} . An alternate route to define the extension L/K is as follows. For every $\sigma \in \Phi$, the character

$$\chi_\sigma: \text{Gal}(K_{\overline{\mathfrak{m}}}/K) \rightarrow \mathbb{Q}^{\text{alg}*}, \quad \chi_\sigma(\mathfrak{a}) = \frac{\sigma \psi'(\mathfrak{a})}{\psi'(\mathfrak{a})}$$

is well-defined via the Artin isomorphism $\text{Gal}(K_{\overline{\mathfrak{m}}}/K) \simeq I(\overline{\mathfrak{m}})/P_1(\overline{\mathfrak{m}})$. Due to the fact that $\bigcap_{\sigma \in \Phi} \ker \chi_\sigma = P_\eta(\overline{\mathfrak{m}})/P_1(\overline{\mathfrak{m}})$, it follows that

$$L = K_{\overline{\mathfrak{m}}}^{\bigcap_{\sigma \in \Phi} \ker \chi_\sigma}.$$

Notice that L/\mathbb{Q} is not necessarily a normal extension; in fact, this is so if and only if $L = \overline{L}$.

Proposition 3.1 *There is an elliptic curve C defined over L such that:*

- (i) $\text{End}_L(C) \simeq \mathcal{O}_K$;
- (ii) its Grossencharacter ψ_C coincides with $\psi' \circ N_{L/K}$;
- (iii) C is isogenous over L to all its $\text{Gal}(L/K)$ -conjugates;
- (iv) the abelian variety A is isogenous over L to the power $C^{[E:K]}$.

Proof. The extreme cases $L = H$ and $L = K_{\overline{\mathfrak{m}}}$ are proved by Gross in [9] and by de Shalit in [5], respectively. For the general case, one can follow the same arguments. Let C_1 be any elliptic curve over L such that $\text{End}_L(C) \simeq \mathcal{O}_K$. Let \mathfrak{n} be its conductor. Once we fix an isomorphism $\theta: K \rightarrow \text{End}_L^0(C_1)$, we can consider the Grossencharacter $\psi_{C_1}: I_L(\mathfrak{n}) \rightarrow K^*$ attached to the pair (C_1, θ) . For a prime ideal \mathfrak{P} of L relatively prime to \mathfrak{n} , we know that $\theta(\psi_{C_1}(\mathfrak{P}))$ is the lifting of the \mathfrak{P} -Frobenius acting on the reduction of $C_1 \bmod \mathfrak{P}$. Recall also that if $\mathfrak{P} \in P_{1,L}(\mathfrak{n})$ then $\psi_{C_1}(\mathfrak{P}) = N_{L/K}(\beta)$, where $\mathfrak{P} = (\beta)$ with $\beta \equiv 1 \pmod{\mathfrak{P}}$.

By class field theory, the composition $\psi' \circ N_{L/K}$ takes values in K^* and the equality $\psi' \circ N_{L/K}(\mathfrak{P}) = N_{L/K}(\beta)$ holds for $\mathfrak{P} = (\beta)$ with $\beta \equiv 1 \pmod{\overline{\mathfrak{m}}\mathcal{O}_L}$. Therefore, the quotient $(\psi' \circ N_{L/K})/\psi_{C_1}$ defines a character $\delta: I_L(\mathfrak{n}\overline{\mathfrak{m}}\mathcal{O}_L)/P_{1,L}(\mathfrak{n}\overline{\mathfrak{m}}\mathcal{O}_L) \rightarrow \mathcal{O}_K^*$ of finite order. The twist $C := C_1 \otimes \delta$ satisfies (i) and (ii). Now, (iii) follows from the fact that $\psi_C = \psi_{C_1}$ for all $\mathfrak{a} \in \text{Gal}(L/K)$ due to (ii).

Now, we check (iv). By Faltings's criterion (for instance, see §2 Corollary 2 of [3]), it suffices to prove that for every prime \mathfrak{P} of L not dividing N nor the conductor of C , the reductions of the abelian varieties A and $C^{\dim A}$ modulo \mathfrak{P} are isogenous over the residue field $\mathcal{O}_L/\mathfrak{P}$. We write $\mathfrak{p}^f = N_{L/K} \mathfrak{P}$, where with no risk of confusion now f is the residue degree of \mathfrak{P} over K . On the one hand, the characteristic polynomial of the endomorphism $\text{Frob}_{\mathfrak{P}}$ acting on the l -adic Tate module of the reduction of A/L modulo \mathfrak{P} , for a prime $l \neq p$, is the characteristic polynomial of the complex representation of $\iota(\psi'(\mathfrak{p}^f))$:

$$P_{A,\mathfrak{P}}(x) = \prod_{\sigma \in \Phi} (x - \sigma \psi'(\mathfrak{p}^f))(x - \overline{\sigma \psi'(\mathfrak{p}^f)}).$$

On the other hand, the corresponding Frobenius characteristic polynomial for C at \mathfrak{P} is

$$P_{C,\mathfrak{P}}(x) = (x - \psi_C(\mathfrak{P}))(x - \overline{\psi_C(\mathfrak{P})}) = (x - \psi'(\mathfrak{p}^f))(x - \overline{\psi'(\mathfrak{p}^f)}).$$

Since $\psi'(\mathfrak{p}^f)$ belongs to K , we obtain $P_{A,\mathfrak{P}}(x) = P_{C,\mathfrak{P}}(x)^{\dim A}$. Thus, A is isogenous over L to $C^{\dim A}$. \square

Proposition 3.2 *The field L is the smallest number field satisfying $\text{End}_{\mathbb{Q}^{\text{alg}}}^0(A) = \text{End}_L^0(A)$.*

Proof. Since A is isogenous over L to the $[E : K]$ -th power of the elliptic curve C , we have $\text{End}_{\mathbb{Q}^{\text{alg}}}^0(A) = \text{End}_L^0(A)$. That L is the smallest number field with this property can be deduced from the following fact. For every $\phi \in \text{End}_L^0(A)$, one has the explicit version of the Skolem-Noether theorem:

$${}^{\mathfrak{p}}\phi = \iota(\psi'(\mathfrak{p})) \cdot \phi \cdot \iota(\psi'(\mathfrak{p}))^{-1},$$

for all $\mathfrak{p} \in I(\overline{\mathfrak{m}})$ not dividing N . To check this equality, it is enough to verify that it holds reduced modulo a prime ideal \mathfrak{P} of L over \mathfrak{p} . The smallest field of definition for all endomorphisms of A is the fixed field L^G , where

$$G = \{v \in \text{Gal}(L/K) : {}^v\phi = \phi \text{ for all } \phi \in \text{End}_L^0(A)\}.$$

By the Čebotarev density theorem, every v in $\text{Gal}(L/K)$ can be written as $v = (\frac{L/K}{\mathfrak{p}})$ for some prime ideal \mathfrak{p} relatively prime to N . We have that $v \in G$ if and only if $\iota(\psi'(\mathfrak{p}))$ is in the center of $\text{End}_L^0(A)$; that is, when $\psi'(\mathfrak{p}) \in K$ and this fact implies that \mathfrak{p} splits completely in L , so that $v = \text{id}$. \square

Let C be an elliptic curve defined over L as in Proposition 3.1. The main theorem of complex multiplication (Theorem 5.4 in [15]) implies the existence of a system of isogenies $\{\mu_{\mathfrak{a}} : C \rightarrow {}^{\mathfrak{a}}C\}$ over L , $(\mathfrak{a}, \overline{\mathfrak{m}}) = 1$, satisfying the following properties:

- (i) $\mu_{\mathfrak{a}\mathfrak{b}} = {}^{\mathfrak{a}}\mu_{\mathfrak{b}} \mu_{\mathfrak{a}}$;
- (ii) if C has good reduction at a prime ideal $\mathfrak{P} \mid \mathfrak{p}$, then $\mu_{\mathfrak{p}}$ is the lifting of the Frobenius map between the reductions of C and ${}^{\mathfrak{p}}C \bmod \mathfrak{P}$.

Attached to the system of isogenies $\{\mu_{\mathfrak{a}}\}$, a one-cocycle can be defined as follows (see also [7]). For a non-zero regular differential ω in $\Omega^1(C/L)$, let $\lambda_{\omega} : I(\overline{\mathfrak{m}}) \rightarrow L^*$ be the map given by

$$\mu_{\mathfrak{a}}^*({}^{\mathfrak{a}}\omega) = \lambda_{\omega}(\mathfrak{a})\omega,$$

where ${}^{\mathfrak{a}}\omega$ denotes the differential in ${}^{\mathfrak{a}}C$ corresponding to ω by conjugation. It follows that λ_{ω} is a one-cocycle, and for all $u \in L^*$ one has

$$\lambda_{u\omega}(\mathfrak{a}) = \lambda_{\omega}(\mathfrak{a}){}^{\mathfrak{a}}u/u.$$

Clearly, the class of λ_{ω} in $H^1(I(\overline{\mathfrak{m}}), L^*)$ does not depend on the particular choice of ω . Note that if $\mathfrak{a} \in P_{\eta}(\overline{\mathfrak{m}})$, then we have $\lambda_{\omega}(\mathfrak{a}) = \psi'(\mathfrak{a})$. The class λ_{ω} in $H^1(I(\overline{\mathfrak{m}}), L^*)$ can be characterized from ψ' as follows:

Proposition 3.3 *Let $\lambda : I(\overline{m}) \rightarrow L^*$ be any one-cocycle satisfying $\lambda(\mathfrak{a}) = \psi'(\mathfrak{a})$ for all $\mathfrak{a} \in I(\overline{m})$ with $(\frac{L/K}{\mathfrak{a}}) = \text{id}$ in $\text{Gal}(L/K)$. Then, $[\lambda] = [\lambda_\omega]$.*

Proof. Assume that $\lambda \in H^1(I(\overline{m}), L^*)$ satisfies $\lambda(\mathfrak{a}) = \psi'(\mathfrak{a})$ for all $\mathfrak{a} \in P_\eta(\overline{m})$. The quotient λ/λ_ω defines a one-cocycle in $H^1(\text{Gal}(L/K), L^*)$. By Hilbert's 90 theorem, we know that there is $u \in L^*$ such that $\lambda(\mathfrak{a})/\lambda_\omega(\mathfrak{a}) = {}^{\mathfrak{a}}u/u$ for all $\mathfrak{a} \in I(\overline{m})$. Thus, we have $[\lambda] = [\lambda_\omega]$. \square

This completes the proof of Theorem 1.2 in the Introduction. From now on, we shall denote by $[A]$ in $H^1(I(\overline{m}), L^*)$ the cohomology class of λ_ω .

4 Modular one-cocycles and elliptic directions

In this section we keep the notations as above and tackle the problem to determine the elliptic directions in $\Omega^1(A)$. The goal is to prove Theorem 1.3 that will be deduced from the next three Propositions after the following

Lemma 4.1 *Let $\pi \in \text{Hom}_L(A, C)$ be a non-constant modular parametrization, and let $\omega \in \Omega^1(C/L)$ be any non-zero regular differential. Denote by*

$$h = \sum_{n \geq 1} \gamma_n q^n \in S_2(\Gamma_1(N))$$

the cusp form associated with the pullback $\pi^(\omega)$. Then,*

(i) $\gamma_1 \in L^*$;

(ii) *for all $\mathfrak{a} \in I(\overline{m})$ relatively prime to N , one has $\mathfrak{a}(\psi'(\mathfrak{a}))^* h = {}^{\mathfrak{a}^{-1}}\lambda_\omega(\mathfrak{a}) {}^{\mathfrak{a}^{-1}}h$;*

(iii) *we have the identity $h = \frac{1}{[L : K]} \sum_{\mathfrak{a} \in \text{Gal}(L/K)} \sum_{\sigma \in \Phi} \frac{{}^{\mathfrak{a}^{-1}}\lambda_\omega(\mathfrak{a})}{{}^\sigma\psi'(\mathfrak{a})} {}^{\mathfrak{a}^{-1}}h$;*

(iv) *$\{\psi'(\mathfrak{a}_i)\}$ is a K -basis of E if and only if $\{{}^{\mathfrak{a}_i^{-1}}h\}$ is an L -basis of $\Omega^1(A/L)$.*

Proof. (i) Since π and ω are defined over L , the cusp form h associated with $\pi^*(\omega)$ has q -expansion $\sum_{n \geq 1} \gamma_n q^n$ with coefficients in L . Since the abelian variety A is simple over K , we have that A is a K -factor of the Weil restriction $\text{Res}_{L/K}(C)$. Thus, the set $\{{}^{\mathfrak{a}}h : \mathfrak{a} \in \text{Gal}(L/K)\}$ generates $\Omega^1(A/L)$. This implies $\gamma_1 \neq 0$.

(ii) It is enough to consider the case when $\mathfrak{a} = \mathfrak{p}$ is a prime ideal not dividing N . Then, the claim follows from the commutativity of the diagram

$$\begin{array}{ccc} A & \xrightarrow{\iota(\psi'(\mathfrak{a}))} & A \\ \mathfrak{a}^{-1}\pi \downarrow & & \downarrow \pi \\ \mathfrak{a}^{-1}C & \xrightarrow{\mathfrak{a}^{-1}\mu_{\mathfrak{a}}} & C \end{array}$$

due to the fact that $\iota(\psi'(\mathfrak{p}))$ and $\mathfrak{p}^{-1}\mu_{\mathfrak{p}}$ are liftings of the corresponding \mathfrak{p} -Frobenius morphisms at a prime ideal $\mathfrak{P} \mid \mathfrak{p}$ of L .

(iii) Write $h = \sum_{v \in \Phi} c_v {}^v f$, with $c_v \in \mathbb{Q}^{\text{alg}}$. By applying (ii), for all $\sigma \in \Phi$ and $\mathfrak{a} \in \text{Gal}(L/K)$, one has

$$\frac{{}^{\mathfrak{a}^{-1}}\lambda_{\omega}(\mathfrak{a})}{{}^{\sigma}\psi'(\mathfrak{a})} {}^{\mathfrak{a}^{-1}}h = \frac{1}{{}^{\sigma}\psi'(\mathfrak{a})} \left(\sum_{v \in \Phi} c_v {}^v \psi'(\mathfrak{a}) {}^v f \right) = \sum_{v \in \Phi} c_v (\chi_v \cdot \chi_{\sigma}^{-1})(\mathfrak{a}) {}^v f.$$

Thus, it holds

$$\sum_{\mathfrak{a}} \sum_{\sigma} \frac{{}^{\mathfrak{a}^{-1}}\lambda_{\omega}(\mathfrak{a})}{{}^{\sigma}\psi'(\mathfrak{a})} {}^{\mathfrak{a}^{-1}}h = \sum_{\sigma, v} \sum_{\mathfrak{a}} c_v (\chi_v \cdot \chi_{\sigma}^{-1})(\mathfrak{a}) {}^v f = [L : K] \sum_v c_v {}^v f = [L : K] h.$$

(iv) If $\{\psi'(\mathfrak{a}_1), \dots, \psi'(\mathfrak{a}_r)\}$ is a K -basis of E , then for every $\mathfrak{a} \in I(\overline{\mathfrak{m}})$ we can write $\psi'(\mathfrak{a}) = \sum_{i=1}^r \alpha_i \psi'(\mathfrak{a}_i)$ with $\alpha_i \in K$. Thus, we obtain

$${}^{\mathfrak{a}^{-1}}\lambda_{\omega}(\mathfrak{a}) {}^{\mathfrak{a}^{-1}}h = \iota(\psi'(\mathfrak{a}))^*(h) = \sum_{i=1}^r \alpha_i \iota(\psi'(\mathfrak{a}_i))^* h = \sum_{i=1}^r \alpha_i {}^{\mathfrak{a}_i^{-1}}\lambda_{\omega}(\mathfrak{a}_i) {}^{\mathfrak{a}_i^{-1}}h.$$

Since $\{{}^{\mathfrak{a}}h : \mathfrak{a} \in \text{Gal}(L/K)\}$ generates $\Omega^1(A/L)$ and $\dim(A) = [E : K]$, it follows that $\{{}^{\mathfrak{a}_1^{-1}}h, \dots, {}^{\mathfrak{a}_r^{-1}}h\}$ is a L -basis of $\Omega^1(A/L)$.

Conversely, assume that $\{{}^{\mathfrak{a}_1^{-1}}h, \dots, {}^{\mathfrak{a}_r^{-1}}h\}$ is a L -basis of $\Omega^1(A/L)$. By using part (ii), if $\sum_{i=1}^r \alpha_i \psi'(\mathfrak{a}_i) = 0$ for some $\alpha_i \in K$, then $\sum_{i=1}^r \alpha_i {}^{\mathfrak{a}_i^{-1}}\lambda_{\omega}(\mathfrak{a}_i) {}^{\mathfrak{a}_i^{-1}}h = 0$. This implies that all $\alpha_i = 0$. Since $\dim(A) = [E : K] = r$, the proof is done. \square

Due to part (i) in the above Lemma 4.1, there is a unique $\omega \in \Omega^1(C/L)$ such that the pullback $\pi^*(\omega)$ gives a normalized cusp form, say

$$h = q + \sum_{n \geq 2} \gamma_n q^n.$$

The particular λ_{ω} will be called *modular with respect to π* or, simply, *π -modular*. For every 1-cocycle $\lambda \in [A]$, we consider the following sums. Let $\sigma \in \Phi$, and set

$$g_{\sigma}(\lambda) := \sum_{\mathfrak{a} \in \text{Gal}(L/K)} \frac{{}^{\mathfrak{a}^{-1}}\lambda(\mathfrak{a})}{{}^{\sigma}\psi'(\mathfrak{a})}.$$

Notice that $g_{\sigma}(\lambda)$ is well-defined and $g_{\sigma}(\lambda) \in {}^{\sigma}E \cdot L$.

Remark 4.1 The sum $g_\sigma(\lambda)$ can be interpreted as a sort of Gauss sum, in the sense that we have

$$g_\sigma(\lambda) = \sum_{\mathfrak{a} \in \text{Gal}(L/K)} \chi_\sigma^{-1}(\mathfrak{a}) u_{\mathfrak{a}}$$

where $u_{\mathfrak{a}} = \mathfrak{a}^{-1} \lambda(\mathfrak{a}) / \psi'(\mathfrak{a})$. If C admits a global minimal Weierstrass equation over L , then the one-cocycle λ attached to a Néron differential satisfies the capitulation property $\lambda(\mathfrak{a}) \mathcal{O}_L = \mathfrak{a} \mathcal{O}_L$ (see remark 10.3 in [7]). Then, $u_{\mathfrak{a}}^e$ is an unit in \mathcal{O}_L^* where e is the order of \mathfrak{a} in $\text{Gal}(L/K)$.

We shall denote the Φ -trace of $g_\sigma(\lambda)$ by

$$\text{tr}_\Phi(\lambda) = \sum_{\sigma \in \Phi} g_\sigma(\lambda) \in L.$$

Remark 4.2 Recall that we have defined $\lambda \in [A]$ to be modular if $\text{tr}_\Phi(\lambda) = [L : K]$ in the Introduction. As it will be shown, both terms (modular and π -modular) turn out to be equivalent.

For every $\gamma \in L^*$ and $\lambda \in [A]$, let λ_γ denote the twisted one-cocycle in $[A]$ given by $\lambda_\gamma(\mathfrak{a}) = \lambda(\mathfrak{a}) \gamma / {}^{\mathfrak{a}}\gamma$. Writing $\lambda = \lambda_\omega$ with some $\omega \in \Omega^1(C/L)$, then $\lambda_\gamma = \lambda_{\frac{1}{\gamma}\omega}$. We shall need the following lemma.

Lemma 4.2 For all $\mathfrak{a} \in I(\overline{\mathfrak{m}})$ and $\sigma \in \Phi$, one has

$$(i) \quad g_\sigma(\lambda_{\mathfrak{a}^{-1}\lambda(\mathfrak{a})}) \cdot \mathfrak{a}^{-1} \lambda(\mathfrak{a}) = g_\sigma(\lambda) \cdot {}^\sigma \psi'(\mathfrak{a});$$

$$(ii) \quad \text{tr}_\Phi(\lambda_{\mathfrak{a}^{-1}\lambda(\mathfrak{a})}) = \mathfrak{a}^{-1} \text{tr}_\Phi(\lambda).$$

Proof. It follows straightforward from the definitions and by using the cocycle relations for λ . \square

Proposition 4.3 Assume that $\lambda \in [A]$ is modular with respect to $\pi \in \text{Hom}_L(A, C)$. Then, $\text{tr}_\Phi(\lambda) = [L : K]$ and

$$h = \frac{1}{[L : K]} \sum_{\sigma \in \Phi} g_\sigma(\lambda) \cdot {}^\sigma f$$

is the normalized elliptic direction in $\pi^*(\Omega^1(C/L))$.

Proof. Since λ is π -modular, there is a non-zero regular differential $\omega \in \Omega^1(C/L)$ such that $\pi^*(\omega)$ is a normalized cusp form $h = q + \sum_{n \geq 2} \gamma_n q^n$ and $\lambda = \lambda_\omega$. By comparing the first Fourier coefficient in the equality at Lemma 4.1 (iii), we have that $\text{tr}_\Phi(\lambda) = [L : K]$. For every $\sigma \in \Phi$, set

$$F_\sigma = \sum_{\mathfrak{b} \in \text{Gal}(L/K)} \frac{{}^{\mathfrak{b}^{-1}} \lambda(\mathfrak{b})}{{}^\sigma \psi'(\mathfrak{b})} {}^{\mathfrak{b}^{-1}} h.$$

Also by Lemma 4.1 (iii), we know that $\sum_{\sigma \in \Phi} F_\sigma = [L : K]h$. From the equality $\iota(\psi'(\mathfrak{a}))^*(\mathfrak{b}^{-1}h) = (\mathfrak{b} \cdot \mathfrak{a})^{-1} \lambda(\mathfrak{a}) (\mathfrak{b} \cdot \mathfrak{a})^{-1} h$, one obtains

$$\begin{aligned} \iota(\psi'(\mathfrak{a}))^*(F_\sigma) &= \sum_{\mathfrak{b} \in \text{Gal}(L/K)} \frac{\mathfrak{b}^{-1} \lambda(\mathfrak{b})}{\sigma \psi'(\mathfrak{b})} (\mathfrak{b} \cdot \mathfrak{a})^{-1} \lambda(\mathfrak{a}) (\mathfrak{b} \cdot \mathfrak{a})^{-1} h \\ &= \sum_{\mathfrak{b} \in \text{Gal}(L/K)} \frac{(\mathfrak{b} \cdot \mathfrak{a})^{-1} \lambda(\mathfrak{b} \cdot \mathfrak{a})}{\sigma \psi'(\mathfrak{b})} (\mathfrak{b} \cdot \mathfrak{a})^{-1} h = {}^\sigma \psi'(\mathfrak{a}) F_\sigma. \end{aligned}$$

Therefore, F_σ and ${}^\sigma f$ differ by a scalar multiple. Since the q -expansion of F_σ begins as $g_\sigma(\lambda)q + \dots$, it follows that $F_\sigma = g_\sigma(\lambda) \cdot {}^\sigma f$, and then $h = \frac{1}{[L : K]} \sum_{\sigma \in \Phi} g_\sigma(\lambda) \cdot {}^\sigma f$. \square

Now, we shall prove that the modular one-cocycles λ in $[A]$ with respect to some modular parametrization π are precisely those that satisfy the trace condition $\text{tr}_\Phi(\lambda) = [L : K]$. To this end, for a given one-cocycle $\lambda \in [A]$ (not necessarily modular), let us consider the K -linear map $\text{pr}: L \rightarrow L$,

$$\text{pr}(u) := \sum_{\mathfrak{a} \in \text{Gal}(L/K)} \left(\sum_{\sigma \in \Phi} \frac{1}{\sigma \psi'(\mathfrak{a})} \right) \mathfrak{a}^{-1} \lambda(\mathfrak{a}) \mathfrak{a}^{-1} u = \begin{cases} u \cdot \text{tr}_\Phi(\lambda_u) & , \text{ if } u \neq 0; \\ 0 & , \text{ otherwise.} \end{cases}$$

Consider the eigenspace $\mathcal{M} = \{u \in L : \text{pr}(u) = [L : K] \cdot u\}$. Notice that λ_u is modular if and only if $u \in \mathcal{M} \setminus \{0\}$. In particular, we know that $\dim_K(\mathcal{M}) > 0$ and it does not depend on the particular choice of $\lambda \in [A]$ used to define the K -linear map pr .

Proposition 4.4 *One has*

- (i) $\text{pr}^2 = [L : K] \text{pr}$;
- (ii) $\dim_K(\mathcal{M}) = [E : K]$;
- (iii) if λ is modular, then $\mathcal{M} = \langle \{\mathfrak{a}^{-1} \lambda(\mathfrak{a})\} \rangle_K$ where \mathfrak{a} runs over $\text{Gal}(L/K)$.

Proof. The first claim comes from the computation:

$$\begin{aligned} \text{pr}^2(u) &= \sum_{\mathfrak{a}} \left(\sum_{\sigma} \frac{1}{\sigma \psi'(\mathfrak{a})} \right) \mathfrak{a}^{-1} \lambda(\mathfrak{a}) \mathfrak{a}^{-1} \left[\sum_{\mathfrak{b}} \left(\sum_{\tau} \frac{1}{\tau \psi'(\mathfrak{b})} \right) \mathfrak{b}^{-1} \lambda(\mathfrak{b}) \mathfrak{b}^{-1} u \right] \\ &= \sum_{\mathfrak{a}} \sum_{\mathfrak{b}} \left(\sum_{\sigma} \frac{1}{\sigma \psi'(\mathfrak{a})} \right) \left(\sum_{\tau} \frac{1}{\tau \psi'(\mathfrak{b})} \right) (\mathfrak{a}\mathfrak{b})^{-1} \lambda(\mathfrak{a}\mathfrak{b}) (\mathfrak{a}\mathfrak{b})^{-1} u \\ &= \sum_{\mathfrak{a}} \sum_{\mathfrak{b}} \left(\sum_{\sigma} \frac{1}{\sigma \psi'(\mathfrak{a})} \right) \left(\sum_{\tau} \frac{1}{\tau \psi'(\mathfrak{a}^{-1}\mathfrak{b})} \right) \mathfrak{b}^{-1} \lambda(\mathfrak{b}) \mathfrak{b}^{-1} u \\ &= \sum_{\mathfrak{b}} \sum_{\mathfrak{a}} \left(\sum_{\sigma, \tau} (\chi_\sigma \chi_\tau^{-1})(\mathfrak{a}) \right) \frac{\mathfrak{b}^{-1} \lambda(\mathfrak{b}) \mathfrak{b}^{-1}}{\tau \psi'(\mathfrak{b})} u \\ &= [L : K] \text{pr}(u). \end{aligned}$$

Let us prove (ii) and (iii) simultaneously. Since $\dim_K(\mathcal{M})$ is independent of the one-cocycle λ chosen in $[A]$, we can (and do) assume that λ is modular. Set

$$h = \frac{1}{[L:K]} \sum_{\sigma \in \Phi} g_\sigma(\lambda) \cdot {}^\sigma f = 1 + \sum_{n>1} \gamma_n q^n.$$

Let $W = \langle \{{}^{\mathfrak{a}^{-1}}\lambda(\mathfrak{a})\}_K$ where \mathfrak{a} runs over $\text{Gal}(L/K)$. We need to show that $W = \mathcal{M}$ and $\dim_K(W) = [E:K]$. Choose $\mathfrak{a}_1, \dots, \mathfrak{a}_r \in I(\overline{\mathfrak{m}})$ such that $\{\psi'(\mathfrak{a}_1), \dots, \psi'(\mathfrak{a}_r)\}$ is a K -basis of E . We claim that $\{{}^{\mathfrak{a}_1^{-1}}\lambda(\mathfrak{a}_1), \dots, {}^{\mathfrak{a}_r^{-1}}\lambda(\mathfrak{a}_r)\}$ is a K -basis of W . Indeed, if $\sum_{i=1}^r \alpha_i {}^{\mathfrak{a}_i^{-1}}\lambda(\mathfrak{a}_i) = 0$ for some α_i in K , then consider $\alpha := \sum_{i=1}^r \alpha_i \psi'(\mathfrak{a}_i) \in E$. It is easy to check that $\iota(\alpha)^*(h) = \sum_{n \geq 1} \gamma'_n q^n$ with $\gamma'_1 = 0$. This forces $\alpha = 0$, since otherwise we get a contradiction from Lemma 4.1 (i) applied to $\iota(\alpha)^*(h)$. Therefore, all $\alpha_i = 0$ which implies that ${}^{\mathfrak{a}_1^{-1}}\lambda(\mathfrak{a}_1), \dots, {}^{\mathfrak{a}_r^{-1}}\lambda(\mathfrak{a}_r)$ are linearly independent. Now, for every ideal $\mathfrak{a} \in I(\overline{\mathfrak{m}})$, one has $\psi'(\mathfrak{a}) = \sum_{i=1}^r \alpha_i \psi'(\mathfrak{a}_i)$ for some $\alpha_i \in K$. By taking q -expansions in the equality

$${}^{\mathfrak{a}^{-1}}\lambda(\mathfrak{a}) {}^{\mathfrak{a}^{-1}}h = \sum_{i=1}^r \alpha_i {}^{\mathfrak{a}_i^{-1}}\lambda(\mathfrak{a}_i) {}^{\mathfrak{a}_i^{-1}}h,$$

we obtain ${}^{\mathfrak{a}^{-1}}\lambda(\mathfrak{a}) = \sum_{i=1}^r \alpha_i {}^{\mathfrak{a}_i^{-1}}\lambda(\mathfrak{a}_i)$. So far, we have $\dim_K(W) = [E:K]$ and the inclusion $W \subseteq \mathcal{M}$ follows from Lemma 4.2 (ii).

To easy notation, set $u_i = {}^{\mathfrak{a}_i^{-1}}\lambda(\mathfrak{a}_i)$ for $1 \leq i \leq r$ and let us show that they generate \mathcal{M} . For any nonzero $u \in \mathcal{M}$, consider the normalized cusp form

$$h_u = \frac{1}{[L:K]} \sum_{\sigma \in \Phi} g_\sigma(\lambda_u) \cdot {}^\sigma f.$$

Since $\{h_{u_1}, \dots, h_{u_r}\}$ is a L -basis of $\Omega^1(A/L)$ by Lemma 4.1 (iv), there are $\gamma_i \in L$ such that $h_u = \sum_{i=1}^r \gamma_i h_{u_i}$. Notice that $\sum_{i=1}^r \gamma_i = 1$. By applying $\iota(\psi'(\mathfrak{a}))^*$ to h_u , and then conjugate by \mathfrak{a} , we obtain

$$\lambda_u(\mathfrak{a}) h_u = \sum_{i=1}^r {}^{\mathfrak{a}}\gamma_i \lambda_{u_i}(\mathfrak{a}) h_{u_i}.$$

Therefore, we have

$$\gamma_i = {}^{\mathfrak{a}}\gamma_i \frac{\lambda_{u_i}(\mathfrak{a})}{\lambda_u(\mathfrak{a})} = {}^{\mathfrak{a}}\gamma_i \frac{{}^{\mathfrak{a}}u}{u} \frac{u_i}{u}$$

for all \mathfrak{a} and $1 \leq i \leq r$. That is, $\beta_i := \gamma_i u / u_i \in K$. Then, $u = \sum_{i=1}^r \beta_i u_i$ since $\sum_{i=1}^r \gamma_i = 1$. The statement (iii) follows. \square

Proposition 4.5 *Let $\lambda' \in [A]$ such that $\text{tr}_\Phi(\lambda') = [L:K]$. Then, λ' is modular with respect to some $\pi' \in \text{Hom}_L(A, C)$.*

Proof. We shall prove that there is $\pi' \in \text{Hom}_L(A, C)$ and $\omega' \in \Omega^1(C/L)$ such that $\pi'^*(\omega')$ corresponds to the normalized cusp form

$$h' = \frac{1}{[L:K]} \sum_{\sigma \in \Phi} \sum_{\mathfrak{a} \in \text{Gal}(L/K)} \frac{\mathfrak{a}^{-1} \lambda'(\mathfrak{a})}{\sigma \psi'(\mathfrak{a})} \cdot \sigma f.$$

Consider any non-constant $\pi \in \text{Hom}_L(A, C)$ and take $\omega \in \Omega^1(C/L)$ such that $\pi^*(\omega)$ corresponds to the normalized cusp form

$$h = \frac{1}{[L:K]} \sum_{\sigma \in \Phi} g_\sigma(\lambda) \cdot \sigma f,$$

where $\lambda = \lambda_\omega$. Let $L = \ker(\text{pr}) \oplus \mathcal{M}$ be the decomposition corresponding to the projector pr attached to λ . Now, there is $\gamma \in \mathcal{M}$ such that $\lambda' = \lambda_\gamma$ and

$$h' = \frac{1}{[L:K]} \sum_{\sigma \in \Phi} g_\sigma(\lambda_\gamma) \cdot \sigma f$$

with $\gamma = \sum_{\mathfrak{a} \in \text{Gal}(L/K)} r_{\mathfrak{a}} \mathfrak{a}^{-1} \lambda(\mathfrak{a})$ for some $r_{\mathfrak{a}} \in K$ due to Proposition 4.4 (iii). We claim that

$$\left(\sum_{\mathfrak{a} \in \text{Gal}(L/K)} r_{\mathfrak{a}} \mathfrak{a}^{-1} \lambda(\mathfrak{a}) \right) h' = \iota \left(\sum_{\mathfrak{a} \in \text{Gal}(L/K)} r_{\mathfrak{a}} \psi'(\mathfrak{a}) \right)^* h. \quad (1)$$

Letting $\Psi = \iota \left(\sum_{\mathfrak{a} \in \text{Gal}(L/K)} r_{\mathfrak{a}} \psi'(\mathfrak{a}) \right) \in \text{End}_K^0(A)$, then it follows

$$h' = \Psi^* \left(\pi^* \left(\frac{1}{\gamma} \omega \right) \right) = (\pi \circ \Psi)^* \left(\frac{1}{\gamma} \omega \right),$$

which implies that λ' is modular. To check (1), we use Lemma 4.2 (i):

$$\begin{aligned} \gamma h' &= \frac{1}{[L:K]} \sum_{\sigma} \sum_{\mathfrak{b}} \frac{\mathfrak{b}^{-1} \lambda(\mathfrak{b}) \mathfrak{b}^{-1} \gamma}{\sigma \psi'(\mathfrak{b})} \sigma f \\ &= \frac{1}{[L:K]} \sum_{\sigma} \sum_{\mathfrak{b}} \sum_{\mathfrak{a}} \frac{\mathfrak{b}^{-1} \lambda(\mathfrak{b}) r_{\mathfrak{a}} (\mathfrak{a}\mathfrak{b})^{-1} \lambda(\mathfrak{a})}{\sigma \psi'(\mathfrak{b})} \sigma f \\ &= \frac{1}{[L:K]} \sum_{\sigma} \sum_{\mathfrak{a}} r_{\mathfrak{a}} g_\sigma(\lambda_{\mathfrak{a}^{-1} \lambda(\mathfrak{a})})^{\mathfrak{a}^{-1}} \lambda(\mathfrak{a}) \sigma f \\ &= \frac{1}{[L:K]} \sum_{\sigma} \sum_{\mathfrak{a}} r_{\mathfrak{a}} \sigma \psi'(\mathfrak{a}) g_\sigma(\lambda) \sigma f \\ &= \frac{1}{[L:K]} \Psi^* \left(\sum_{\sigma} g_\sigma(\lambda) \sigma f \right) = \Psi^*(h). \quad \square \end{aligned}$$

The transitivity of the action of $\iota(E^*)$ on the set of elliptic directions follows from the equality (1). To finish the proof of Theorem 1.3, it remains to determine the q -expansions of the normalized elliptic directions. For it, first we need a technical lemma.

Lemma 4.6 Let $\ell: I(\mathfrak{m}) \rightarrow L^*$ be a map such that $\ell(\mathfrak{a}) = \psi(\mathfrak{a})$ for all $\mathfrak{a} = \text{id}$ in $\text{Gal}(\bar{L}/K)$. Let $\tau: \text{Gal}(\bar{L}/K) \rightarrow \text{Gal}(L/K)$ be a map such that $\ell(\mathfrak{a}\mathfrak{b}) = \ell(\mathfrak{a})^{\tau(\mathfrak{a})}\ell(\mathfrak{b})$ for all $\mathfrak{a} \in I(\mathfrak{m})$. Then, the identity

$$\frac{1}{[L:K]} \sum_{\sigma \in \Phi} \beta_{\sigma}^{\sigma} \psi(\mathfrak{c}) = \ell(\mathfrak{c}) \quad (2)$$

holds for all $\mathfrak{c} \in I(\mathfrak{m})$ if and only if

$$\beta_{\sigma} = \sum_{\mathfrak{a} \in \text{Gal}(\bar{L}/K)} \frac{\ell(\mathfrak{a})}{\sigma \psi(\mathfrak{a})} \quad \text{and} \quad \sum_{\sigma \in \Phi} \beta_{\sigma} = [L:K]. \quad (3)$$

Proof. Assume (3). For every $\mathfrak{c} \in I(\mathfrak{m})$, we have

$$\begin{aligned} \sum_{\sigma \in \Phi} \left(\sum_{\mathfrak{a} \in \text{Gal}(\bar{L}/K)} \frac{\ell(\mathfrak{a})}{\sigma \psi(\mathfrak{a})} \right) \sigma \psi(\mathfrak{c}) &= \sum_{\sigma \in \Phi} \left(\sum_{\mathfrak{a} \in \text{Gal}(\bar{L}/K)} \frac{\ell(\mathfrak{a}\mathfrak{c})}{\sigma \psi(\mathfrak{a}\mathfrak{c})} \right) \sigma \psi(\mathfrak{c}) \\ &= \ell(\mathfrak{c}) \sum_{\sigma \in \Phi} \left(\sum_{\mathfrak{a} \in \text{Gal}(\bar{L}/K)} \frac{\tau(\mathfrak{c}) \ell(\mathfrak{a})}{\sigma \psi(\mathfrak{a})} \right) \\ &= \ell(\mathfrak{c})^{\tau(\mathfrak{c})} \left(\sum_{\mathfrak{a} \in \text{Gal}(\bar{L}/K)} \ell(\mathfrak{a}) \left(\sum_{\sigma \in \Phi} \frac{1}{\sigma \psi(\mathfrak{a})} \right) \right) \\ &= \ell(\mathfrak{c})^{\tau(\mathfrak{c})} \left(\sum_{\sigma \in \Phi} \beta_{\sigma} \right) = \ell(\mathfrak{c}) [L:K]. \end{aligned}$$

Now, suppose (2). Fix $\nu \in \Phi$. Note that for $\sigma \in \Phi$, the characters χ_{σ} and χ_{ν} are equal if and only if $\sigma = \nu$. For every $\mathfrak{a} \in \text{Gal}(\bar{L}/K)$, one has:

$$\frac{\ell(\mathfrak{a})}{\nu \psi(\mathfrak{a})} = \frac{1}{[L:K]} \left(\beta_{\nu} + \sum_{\sigma \in \Phi \setminus \{\nu\}} \beta_{\sigma} \frac{\sigma \psi(\mathfrak{a})}{\nu \psi(\mathfrak{a})} \right) = \frac{1}{[L:K]} \left(\beta_{\nu} + \sum_{\sigma \in \Phi \setminus \{\nu\}} \beta_{\sigma} (\chi_{\sigma} \chi_{\nu}^{-1})(\mathfrak{a}) \right).$$

Summing over all \mathfrak{a} , then

$$\sum_{\mathfrak{a} \in \text{Gal}(\bar{L}/K)} \frac{\ell(\mathfrak{a})}{\nu \psi'(\mathfrak{a})} = \beta_{\nu} + \frac{1}{[L:K]} \left(\sum_{\sigma \in \Phi \setminus \{\nu\}} \beta_{\sigma} \sum_{\mathfrak{a} \in \text{Gal}(\bar{L}/K)} (\chi_{\sigma} \chi_{\nu}^{-1})(\mathfrak{a}) \right) = \beta_{\nu}.$$

The condition $\sum_{\sigma \in \Phi} \beta_{\sigma} = [L:K]$ is obtained by replacing \mathfrak{a} with \mathcal{O} in (2). \square

Proposition 4.7 Assume that $\lambda \in [A]$ satisfies $\text{tr}_{\Phi}(\lambda) = [L:K]$. Consider the normalized cusp form

$$h = \frac{1}{[L:K]} \sum_{\sigma \in \Phi} g_{\sigma}(\lambda) \cdot \sigma f.$$

Then,

(i) one has

$$h = \begin{cases} \sum_{(\mathfrak{a}, \mathfrak{m})=1} \mathfrak{a}^{-1} \lambda(\mathfrak{a}) q^{N(\mathfrak{a})} & , \text{ if } K \not\subseteq E_f; \\ \sum_{(\mathfrak{a}, \mathfrak{m})=1} \frac{N(\mathfrak{a})}{\lambda(\bar{\mathfrak{a}})} q^{N(\mathfrak{a})} & , \text{ if } K \subseteq E_f; \end{cases}$$

(ii) for all $\mathfrak{c} \in I(\bar{\mathfrak{m}})$, we have $\iota(\psi'(\mathfrak{c}))^*(h) = \mathfrak{c}^{-1} \lambda(\mathfrak{c}) \mathfrak{c}^{-1} h$.

Proof. For all $\mathfrak{a} \in I(\mathfrak{m})$, set

$$\ell(\mathfrak{a}) = \begin{cases} \mathfrak{a}^{-1} \lambda(\mathfrak{a}) & , \text{ if } K \not\subseteq E_f; \\ \frac{N(\mathfrak{a})}{\lambda(\bar{\mathfrak{a}})} & , \text{ if } K \subseteq E_f. \end{cases}$$

It is clear that $\ell(\mathfrak{a}\mathfrak{b})$ is $\ell(\mathfrak{a})\mathfrak{a}^{-1}\ell(\mathfrak{b})$ or $\ell(\mathfrak{a})\bar{\mathfrak{a}}\ell(\mathfrak{b})$ depending on whether $K \not\subseteq E_f$ or not, respectively. Since for the case $K \subseteq E_f$ one has

$$\frac{\ell(\mathfrak{a}^{-1})}{\sigma\psi(\mathfrak{a}^{-1})} = \frac{\bar{\mathfrak{a}}^{-1}(1/\ell(\mathfrak{a}))}{\sigma\psi(\mathfrak{a}^{-1})} = \frac{\bar{\mathfrak{a}}^{-1}(N(\mathfrak{a})/\ell(\mathfrak{a}))}{N(\mathfrak{a})/\sigma\psi(\mathfrak{a})} = \frac{\bar{\mathfrak{a}}^{-1}\lambda(\bar{\mathfrak{a}})}{\sigma\psi'(\bar{\mathfrak{a}})},$$

for all $\sigma \in \Phi$, then in both cases it follows that $g_\sigma(\lambda) = \sum_{\mathfrak{a} \in \text{Gal}(\bar{L}/K)} \ell(\mathfrak{a})/\sigma\psi(\mathfrak{a})$. By using Lemma 4.6, a case-by-case computation shows that for all $\mathfrak{a} \in I(\mathfrak{m})$ and $\mathfrak{c} \in I(\bar{\mathfrak{m}})$ one has

$$\frac{1}{[L:K]} \sum_{\sigma \in \Phi} g_\sigma(\lambda) \sigma\psi(\mathfrak{a}) \sigma\psi'(\mathfrak{c}) = \mathfrak{c}^{-1} \lambda(\mathfrak{c}) \mathfrak{c}^{-1} \ell(\mathfrak{a}). \quad (4)$$

Plugging $\mathfrak{c} = 1$ in (4) it follows part (i). Part (ii) follows from part (i) and (4). \square

Now, Theorem 1.3 in the Introduction follows from Propositions 4.3, 4.5 and 4.7. Note that due to Proposition 4.4, all one-cocycles in $[A]$ are modular if and only if $[E:K] = [L:K]$; i.e., when A is K -isogenous to $\text{Res}_{L/K}(C)$. In general, in order to determine a modular one-cocycle in $[A]$ a strategy emerges from the previous results. Indeed, first one can build a one-cocycle $\lambda \in [A]$ by solving and combining norm equations. If $\text{tr}_\Phi(\lambda) \neq 0$, then $\lambda_{\text{tr}_\Phi(\lambda)}$ is modular since its Φ -trace equals $[L:K]$. Alternatively, if $\text{tr}_\Phi(\lambda) = 0$ or in any circumstance, the nullspace of the K -linear map $\text{pr} - [L:K] \text{Id}$ provides all $u \in L$ such that λ_u is modular.

We also remark that for the case $K \subseteq E_f$, there are elliptic quotients of A_f that do not factor through neither A nor \bar{A} . These quotients can be obtained using the above results plus the Weil involution acting on A_f .

We conclude this section with three open questions: one concerning about the isomorphism $\iota: E \rightarrow \text{End}_K^0(A)$ and the others about the elliptic optimal quotients of A . All the results of the paper hold when we replace $J_1(N)$ with $\text{Jac}(X_\Gamma)$, where Γ is an intermediate congruence subgroup between $\Gamma_1(N)$ and $\Gamma_0(N)$ such that $f \in S_2(\Gamma)$

and X_Γ is the modular curve attached to this subgroup. Although the optimal quotient A of $A_f^{(\Gamma)}$ does depend on Γ , it is known that $\iota(T_p) \in \text{End}_{\mathbb{Q}}(A_f^{(\Gamma)})$ and, thus, $\iota(T_p)$ belongs to $\text{End}_K(A)$ for all Γ .

Question 4.8 *Is $\iota(\psi(\mathfrak{a})) \in \text{End}_K(A)$ for all integral ideals \mathfrak{a} and all Γ ?*

We ask ourselves whether the j -invariants of optimal modular parametrizations of CM elliptic curves are not far from being also *optimal* in the sense of having CM by the maximal order of K . Of course, if $\iota(\mathcal{O}_K) \subset \text{End}_K(A)$ all optimal elliptic quotients have multiplication by \mathcal{O}_K . If $\iota(\eta(\mathfrak{a})) \in \text{End}_K(A)$ for all integral ideals $\mathfrak{a} \in I(\overline{\mathfrak{m}})$, then the j -invariants of all optimal elliptic quotients are in the Hilbert class field H . From Cremona's tables ($N < 130000$), we have checked that all optimal elliptic quotients over \mathbb{Q} with CM of $J_0(N)$ have complex multiplication by \mathcal{O}_K . Also, the same experimental result has been obtained in all examples over \mathbb{Q}^{alg} collected by the authors.

Question 4.9 *Assume that $\pi \in \text{Hom}_L(A, C)$ is optimal. Does C have complex multiplication by \mathcal{O}_K ?*

And the last question is related to the above Remark 4.1.

Question 4.10 *Is it true that the existence of an optimal elliptic quotient of A having global minimal model over L is equivalent to the existence of a modular one-cocycle $\lambda \in [A]$ with values $\lambda(\mathfrak{a})$ in the ring of integers \mathcal{O}_L for all integral ideals $\mathfrak{a} \in I(\overline{\mathfrak{m}})$?*

In the next sections, we apply the above results and focus our attention on Gross's elliptic curves $A(p)$. We also give a positive answer to the second question mentioned above for the particular case of level $N = p^2$.

5 CM elliptic optimal quotients of $J_1(p^2)$

In the sequel p is a prime > 3 and such that $p \equiv 3 \pmod{4}$. The discriminant of $K = \mathbb{Q}(\sqrt{-p})$ is $-p$. Set $\mathfrak{p} = \sqrt{-p} \mathcal{O}_K$. Let \mathcal{X} denote the set of Hecke characters mod \mathfrak{p} and let \mathcal{Y} be the set of Dirichlet characters $\eta : (\mathcal{O}_K/\mathfrak{p})^* \rightarrow \mathbb{C}^*$ such that $\eta(-1) = -1$.

To every Hecke character $\psi \in \mathcal{X}$, we attach its eta-character η in \mathcal{Y} defined as in Section 3 by $\eta(a) = \psi((a))/a$, and it can be easily seen that this map $\mathcal{X} \rightarrow \mathcal{Y}$ is surjective. The Nebentypus $\varepsilon : (\mathbb{Z}/p\mathbb{Z})^* \rightarrow \mathbb{C}^*$ of the newform $f \in S_2(\Gamma_1(p^2))$ associated with ψ is given by $\varepsilon(n) = \chi(n)\eta(n)$, where χ is the quadratic Dirichlet character associated with K . In this case, we have that $\text{ord } \varepsilon = (\text{ord } \eta)/2$.

By the results in Section 3, we know that the elliptic optimal quotients of the abelian variety A_f are defined over a number field L , which is a cyclic extension of H of degree $\text{ord } \varepsilon$ contained in $K_{\mathfrak{p}}$.

Proposition 5.1 *The ray class field $K_{\mathfrak{p}}$ satisfies $[K_{\mathfrak{p}} : H] = (p-1)/2$ and we have $K_{\mathfrak{p}} = H \cdot \mathbb{Q}(\zeta_p)$, where $\zeta_p = e^{2\pi i/p}$.*

Proof. From the exact sequence

$$1 \longrightarrow (\mathcal{O}_K/\mathfrak{p})^* / \mathcal{O}_K^* \longrightarrow I(\mathfrak{p})/P_1(\mathfrak{p}) \longrightarrow I(\mathcal{O}_K)/P(\mathcal{O}_K) \longrightarrow 1,$$

we know that the Galois group $\text{Gal}(K_{\mathfrak{p}}/H)$ is isomorphic to $(\mathcal{O}_K/\mathfrak{p})^* / \mathcal{O}_K^*$ and, thus, one has $[K_{\mathfrak{p}} : H] = (p-1)/2$. Consider the morphism $\Phi_{\mathfrak{p}} : I(\mathfrak{p}) \rightarrow \text{Gal}(H \cdot \mathbb{Q}(\zeta_p)/K)$ given by the Artin symbol. We claim that $\Phi_{\mathfrak{p}}$ has kernel $P_1(\mathfrak{p})$, which implies that $K_{\mathfrak{p}} \subseteq H \cdot \mathbb{Q}(\zeta_p)$. Indeed, for any ideal $\mathfrak{a} \in I(\mathfrak{p})$, we have that $\Phi_{\mathfrak{p}}(\mathfrak{a})$ acts trivially on H if and only if $\mathfrak{a} \in P(\mathfrak{p})$, that is $\mathfrak{a} = a\mathcal{O}$. Moreover, $\Phi_{\mathfrak{p}}(a\mathcal{O})$ acts trivially on $\mathbb{Q}(\zeta_p)$ if and only if the Artin symbol $\left(\frac{\mathbb{Q}(\zeta_p)/\mathbb{Q}}{N(a)}\right)$ is the identity; i.e., $N(a) \equiv 1 \pmod{\mathfrak{p}}$ which is equivalent to $\mathfrak{a} \in P_1(\mathfrak{p})$ since $N(a) \equiv a^2 \pmod{\mathfrak{p}}$. Finally, for any subfield F of $\mathbb{Q}(\zeta_p)$ which contains K we have that $H \cap F = K$ since either $F = K$ or F/K is ramified at \mathfrak{p} . Hence, one has the equality $[H \cdot \mathbb{Q}(\zeta_p) : H] = (p-1)/2 = [K_{\mathfrak{p}} : H]$ and the statement follows. \square

We shall need the following lemma.

Lemma 5.2 *Let $\psi \in \mathcal{X}$ and denote by η and f its eta-character and newform, respectively. Then,*

(i) *for every ideal $\mathfrak{a} \in I(\mathfrak{p})$, one has*

$$\text{Tr}_{E/K}(\psi(\mathfrak{a})) = \begin{cases} a \sum_{\sigma \in \Phi} \sigma \eta(a) & , \text{ if } \mathfrak{a} = a\mathcal{O}_K; \\ 0 & , \text{ if } \mathfrak{a} \notin P(\mathfrak{p}); \end{cases}$$

(ii) *let η' and f' denote the eta-character and newform associated with $\psi' \in \mathcal{X}$. Then, $f' = {}^{\sigma}f$ for some $\sigma \in \text{Gal}(\mathbb{Q}^{\text{alg}}/K)$ if and only if $\ker \eta' = \ker \eta$.*

Proof. First, let us prove (i). When $\mathfrak{a} = a\mathcal{O}_K$, the claim on the trace is clear since ${}^{\sigma}\psi(a) = a^{\sigma} \eta(a)$. Suppose that $\mathfrak{a} \notin P(\mathfrak{p})$, and let n be the order of \mathfrak{a} in $I(\mathfrak{p})/P(\mathfrak{p})$. Notice that $n > 1$ and $\psi(\mathfrak{a}) \notin K$. For every $\sigma \in \Phi$, we have ${}^{\sigma}\psi(\mathfrak{a}) = \psi(\mathfrak{a})\zeta_{\sigma}$ for some $\zeta_{\sigma} \in \mu_n$, where μ_n denotes the group of n th roots of unity. Thus, we have

$$\sum_{\sigma \in \Phi} {}^{\sigma}\psi(\mathfrak{a}) = \psi(\mathfrak{a}) \sum_{\sigma \in \Phi} \zeta_{\sigma} \in K.$$

Therefore, either $\text{Tr}_{E/K}(\psi(\mathfrak{a})) = 0$ or $\psi(\mathfrak{a}) \in K(\mu_n)$. Let us see that the last possibility does not occur. For it, assume that $\psi(\mathfrak{a}) \in K(\mu_n)$ which implies that the extension $K(\psi(\mathfrak{a}))/K$ is normal. Since n is the minimum positive integer such that $\psi(\mathfrak{a})^n \in K$, it follows that either $\mu_n \subset K$ or $\psi(\mathfrak{a})^{2n} \in K^n$ (see Proposition 2 in [14]). Since $\psi(\mathfrak{a}) \notin K$,

we must have that $\psi(\mathfrak{a}^{2n}) = b^n = \psi((b\mathcal{O}_K)^n)$ for some $b \in K$ and, hence, $\mathfrak{a}^2 = b\mathcal{O}_K$. The class number of K being odd, we get a contradiction.

Let us prove (ii). If $f' = {}^\sigma f$ for some $\sigma \in \text{Gal}(\mathbb{Q}^{\text{alg}}/K)$ then the statement is clear since $\eta' = {}^\sigma \eta$. Now, suppose that $\ker \eta' = \ker \eta$. We claim that

$$\{{}^\sigma f: \sigma \in \Phi\} \cap \{{}^\sigma f': \sigma \in \Phi'\} \neq \emptyset,$$

where Φ' is the corresponding set of K -embeddings $\mathbb{Q}(\psi') \hookrightarrow \mathbb{C}$. Let us consider the normalized cusp forms

$$h = \frac{1}{|\Phi|} \sum_{\sigma \in \Phi} {}^\sigma f = q + \dots,$$

$$h' = \frac{1}{|\Phi'|} \sum_{\sigma \in \Phi'} {}^\sigma f' = q + \dots$$

in $S_2(\Gamma_1(p^2))^{\text{new}}$. Since $K \not\subseteq \mathbb{Q}(\text{im } \eta)$ and $\ker \eta' = \ker \eta$, there is $\tau \in \Phi$ such that ${}^\tau \eta(a) = \eta'(a)$ for all $a \in \mathcal{O}_K$ coprime with \mathfrak{p} . By applying (i), we obtain the equality

$$h = \sum_{\mathfrak{a} \in P(\mathfrak{p})} \frac{\text{Tr}_{E/K}(\psi(\mathfrak{a}))}{|\Phi|} q^{N(\mathfrak{a})} = \sum_{\mathfrak{a} \in P(\mathfrak{p})} \frac{\text{Tr}_{E/K}(\psi'(\mathfrak{a}))}{|\Phi'|} q^{N(\mathfrak{a})} = h'.$$

Therefore, the \mathbb{Q}^{alg} -vector spaces generated by $\{{}^\sigma f: \sigma \in \Phi\}$ and $\{{}^\sigma f': \sigma \in \Phi'\}$ have a common non-zero cusp form, which implies that $f' = {}^\sigma f$ for some $\sigma \in \text{Gal}(\mathbb{Q}^{\text{alg}}/\mathbb{Q})$ (cf. Proposition 3.2 in [1]). Since $h \in \langle {}^\tau f: \tau \in \text{Gal}(\mathbb{Q}^{\text{alg}}/K) \rangle \cap \langle {}^\tau f': \tau \in \text{Gal}(\mathbb{Q}^{\text{alg}}/K) \rangle$, it follows that $\sigma \in \text{Gal}(\mathbb{Q}^{\text{alg}}/K)$. \square

Proposition 5.3 *For every positive divisor d of $(p-1)/2$ there is a unique abelian variety A_f of CM elliptic type of level p^2 such that the Nebentypus of f has order d ; this abelian variety satisfies that $K \not\subseteq E_f$ and $\dim A_f = [H:K]\varphi(d)$, where φ is the Euler function.*

Proof. Let d be a divisor of $(p-1)/2$ and take $\psi \in \mathcal{X}$ such that its eta-character has order $2d$. Let us denote by f the newform attached to ψ , whose Nebentypus ε has order d . First, let us show that $K \not\subseteq E_f$. Indeed, let $\psi_c \in \mathcal{X}$ defined by $\psi_c(\mathfrak{a}) = \overline{\psi(\overline{\mathfrak{a}})}$. The eta-character and the normalized newform attached to ψ_c are clearly $\overline{\eta}$ and \overline{f} , respectively. Since $\ker \overline{\eta} = \ker \eta$, Lemma 5.2 (ii) ensures that $\overline{f} \in \{{}^\sigma f: \sigma \in \Phi\}$, which implies $K \not\subseteq E_f$. The same argument can be applied to another newform f' obtained from $\psi' \in \mathcal{X}$ whose associated character η' has order $2d$ to show that f' belongs to $\{{}^\sigma f: \sigma \in \Phi\}$, which proves that A_f is unique when the order of ε has been fixed.

Since $K \not\subseteq E_f$, the equality $\dim A_f = [E_f: \mathbb{Q}] = [E: K]$ holds. Now, we have that $[E: K] = |\{{}^\sigma f: \sigma \in \Phi\}| = |\{{}^\sigma \psi: \sigma \in \Phi\}|$. Again using part (ii) of Lemma 5.2, we obtain

$$[E: K] = |\{\sigma \in \Phi: \eta = {}^\sigma \eta\}| \cdot |\{{}^\sigma \eta: \sigma \in \Phi\}| = |\{\sigma \in \Phi: \eta = {}^\sigma \eta\}| \cdot \varphi(d).$$

Since the condition ${}^\sigma \eta = \eta$ is equivalent to $\psi/{}^\sigma \psi$ being a character of $\text{Gal}(H/K)$, it follows $\dim A_f = [H: K]\varphi(d)$. \square

Remark 5.1 *Note that the number of abelian varieties A_f of CM elliptic type of level p^2 is the number of divisors of $(p-1)/2$. Also for every number field L intermediate between H and $H \cdot \mathbb{Q}(\zeta_p)$ there is a unique abelian variety A_f of CM elliptic type and level p^2 for which L is its splitting field as defined in Section 3.*

Next, in order to show that the CM elliptic optimal quotients of A_f in $J_1(p^2)$ have endomorphism ring isomorphic to \mathcal{O}_K , we shall need to use some auxiliary congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ of level p^2 . To this end, fix f a newform in $S_2(\Gamma_1(p^2))$ attached to a Hecke character $\psi \in \mathcal{X}$. Let ε denote the Nebentypus of f . Let us consider the following congruence subgroups of level p^2 :

$$\Gamma_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p^2) : a \equiv d \equiv 1 \pmod{p} \right\},$$

and Γ_ε as in the introduction; i.e.,

$$\Gamma_\varepsilon = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p^2) : \varepsilon(d) = 1 \right\}.$$

It is clear that $\Gamma_1(p^2) \subseteq \Gamma_p \subseteq \Gamma_\varepsilon$ and $f \in S_2(\Gamma_\varepsilon)$. For any intermediate congruence subgroup Γ of level p^2 satisfying $\Gamma_1(p^2) \subseteq \Gamma \subseteq \Gamma_\varepsilon$, let X_Γ be the modular curve over \mathbb{Q} attached to Γ . We shall denote by $A_f^{(\Gamma)}$ the optimal quotient of the jacobian of X_Γ attached to f by Shimura. More precisely, let I_f be the annihilator of f in the Hecke algebra acting on $\mathrm{Jac}(X_\Gamma)$. Then,

$$A_f^{(\Gamma)} = \mathrm{Jac}(X_\Gamma) / I_f(\mathrm{Jac}(X_\Gamma)).$$

Proposition 5.4 *Let f and Γ be as above. Then, all elliptic optimal quotients of $A_f^{(\Gamma)}$ have complex multiplication by \mathcal{O}_K .*

Proof. Fix an elliptic direction in $\Omega^1(A_f)$ and let C_Γ be an elliptic optimal quotient attached to this direction. By Proposition 5.3 and Theorem 1.2, we know that $K \not\subseteq E_f$ and thus all endomorphisms of $A_f^{(\Gamma)}$ are defined over its splitting field, say L , that satisfies $L \subseteq K_p$. Let c_Γ denote the conductor of the order $\mathcal{O}_\Gamma \simeq \mathrm{End}_L(C_\Gamma)$ in \mathcal{O}_K . We want to show that $c_\Gamma = 1$, and split the proof in three steps.

Step 1: $c_\Gamma \mid 2$ for all Γ . Since $\mathrm{End}(C_\Gamma) = \mathrm{End}_L(C_\Gamma)$, one has that L contains the ring class field of \mathcal{O}_Γ , say K_Γ . Notice that $K_\Gamma \subseteq L \subseteq K_p$. But $p \nmid c_\Gamma$, since otherwise p must divide $[L : H]$ (cf. Proposition 7.24 in [4]) and this degree is a divisor of $(p-1)/2$. Hence, K_Γ is an unramified extension of the Hilbert class field and, therefore, it must coincide with H . Again by Proposition 7.24 in [4], we obtain that $c_\Gamma \mid 2$.

Step 2: c_Γ does not depend on Γ . We consider the natural projection $\pi : X_\Gamma \rightarrow X_{\Gamma_\varepsilon}$. The degree of π is odd since it divides $[\Gamma_1(p^2) : \Gamma_0(p^2)]/\{\pm 1\} = p(p-1)/2$ and $p \equiv 3 \pmod{4}$.

Let $\pi_{\Gamma, \Gamma_\varepsilon} : \text{Jac}(X_\Gamma) \longrightarrow A_{\Gamma, \Gamma_\varepsilon}$ be the optimal quotient over \mathbb{Q} for which there is an isogeny $v : A_{\Gamma, \Gamma_\varepsilon} \longrightarrow \text{Jac}(X_{\Gamma_\varepsilon})$ defined over \mathbb{Q} rending the following diagram

$$\begin{array}{ccc} \text{Jac}(X_\Gamma) & \xrightarrow{\pi_*} & \text{Jac}(X_{\Gamma_\varepsilon}) \\ & \searrow \pi_{\Gamma, \Gamma_\varepsilon} \quad \nearrow v & \\ & A_{\Gamma, \Gamma_\varepsilon} & \end{array}$$

commutative. Since every element of the group $H_1(X_{\Gamma_\varepsilon}, \mathbb{Z})/\pi_*(H_1(X_\Gamma, \mathbb{Z}))$ has order dividing $\deg \pi$, the cardinality of this group is odd. From the group isomorphism $\ker v \simeq H_1(X_{\Gamma_\varepsilon}, \mathbb{Z})/\pi_*(H_1(X_\Gamma, \mathbb{Z}))$, it follows that $\deg v$ is odd. Since $A_f^{(\Gamma)}$ is an optimal quotient of $A_{\Gamma, \Gamma_\varepsilon}$, there is an isogeny $v_f : A_f^{(\Gamma)} \longrightarrow A_f^{(\Gamma_\varepsilon)}$ whose degree divides $\deg v$. Hence, for every optimal elliptic quotient $\pi_\Gamma : A_f^{(\Gamma)} \longrightarrow C_\Gamma$ there is an optimal elliptic quotient $\pi_\varepsilon : A_f^{(\Gamma_\varepsilon)} \longrightarrow C_{\Gamma_\varepsilon}$ and an isogeny $\mu : C_\Gamma \longrightarrow C_{\Gamma_\varepsilon}$ rending the diagram

$$\begin{array}{ccc} A_f^{(\Gamma)} & \xrightarrow{v_f} & A_f^{(\Gamma_\varepsilon)} \\ \pi_\Gamma \downarrow & & \downarrow \pi_\varepsilon \\ C_\Gamma & \xrightarrow{\mu} & C_{\Gamma_\varepsilon} \end{array}$$

commutative. It is clear that $\deg \mu$ is odd since it divides $\deg v_f$. Thus c_{Γ_ε} and c_Γ can only differ by an odd factor, which implies that c_Γ is independent of the group Γ .

Step 3: $c_\Gamma = 1$ for all Γ . Now, it suffices to prove $c_\Gamma = 1$ for a particular subgroup Γ . We consider $\Gamma = \Gamma_p$. Following Shimura in [17], we know that the matrix

$$\begin{pmatrix} 1 & 1/p \\ 0 & 1 \end{pmatrix}$$

lies in the normalizer of Γ_p in $\text{SL}_2(\mathbb{R})$ and provides an automorphism u of X_{Γ_p} of order p . Set

$$G = \sum_{\substack{1 \leq i < p \\ \chi(i)=1}} (u^*)^i \in \text{End Jac}(X_{\Gamma_p}).$$

We claim that G leaves stable the subvariety $I_f(\text{Jac}(X_{\Gamma_p}))$, which is equivalent to saying that G leaves stable the vector space generated by the set of eigenforms in $S_2(\Gamma_p)$ which are not Galois conjugates of f . In fact, the action of G on all eigenforms of $S_2(\Gamma_p)$ can be described as follows. It is well-known that if we denote by New_Γ the set of normalized newforms in $S_2(\Gamma)$, then the set of normalized eigenforms in $S_2(\Gamma_p)$ is the disjoint union of New_{Γ_p} , \mathcal{S}_1 , and \mathcal{S}_2 , where $\mathcal{S}_1 = \text{New}_{\Gamma_1(p)} \cap$

$S_2(\Gamma_p)$, $\mathcal{S}_2 = B_p(\text{New}_{\Gamma_1(p)}) \cap S_2(\Gamma_p)$, and B_p is the operator acting as $B_p(h(q)) = h(q^p)$. With $\zeta_p = e^{2\pi i/p}$ and from the equality

$$\sum_{\substack{1 \leq i < p \\ \chi(i)=1}} \zeta_p^i = \frac{-1 + \sqrt{-p}}{2},$$

it can be easily checked that every eigenform $h(q) = \sum_{n \geq 1} b_n q^n \in S_2(\Gamma_p)$ satisfies:

$$G^*(h) = \begin{cases} \frac{-1 + \sqrt{-p}}{2} h + \frac{p - \sqrt{-p}}{2} b_p B_p(h) & , \text{ if } h \in \text{New}_{\Gamma_p} \cup \mathcal{S}_1, \\ \frac{p-1}{2} h & , \text{ if } h \in \mathcal{S}_2. \end{cases}$$

The claim follows from the fact that all $h \in \text{New}_{\Gamma_p}$ have level p^2 and Nebentypus whose conductor divides p and, thus, $b_p = 0$ (see subsection 1.8 in [6]).

Since G leaves stable the subvariety $I_f(\text{Jac}(X_{\Gamma_p}))$, then G induces an endomorphism of $A_f^{(\Gamma_p)}$, which we still denote by G . Due to the fact that G acts on $\Omega^1(A_f^{(\Gamma_p)})$ as the multiplication by $(-1 + \sqrt{-p})/2$, it follows that G leaves stable all subvarieties of $A^{(\Gamma_p)}$. Thus, $(-1 + \sqrt{-p})/2 \in \mathcal{O}_{\Gamma_p}$ and the statement follows. \square

As for Gross's elliptic curves, we obtain the following result, which concludes the proof of Theorem 1.4.

Corollary 5.5 *Let f be a CM normalized newform with trivial Nebentypus. The elliptic curve $A(p)$ and its Galois conjugates are the optimal quotients of $A_f^{(\Gamma)}$ over the Hilbert class field H , for all subgroups Γ with $\Gamma_1(p^2) \subseteq \Gamma \subseteq \Gamma_0(p^2)$.*

Proof. By Theorem 20.1 in [9], we know that $A(p)$ is a quotient of $J_0(p^2)$ defined over H , attached to a newform f with trivial Nebentypus. Notice that the corresponding field L coincides with the Hilbert class field H . Since we have $K \not\subseteq E_f$, by Theorems 1.1 and 1.2, every elliptic optimal quotient C_Γ of $A_f^{(\Gamma)}$ is defined over H and the abelian variety $A_f^{(\Gamma)}$ is simple over K . Since $\dim A_f^{(\Gamma)} = [H : K]$, it follows that $A_f^{(\Gamma)}$ is K -isogenous to the Weil restriction $\text{Res}_{H/K} C_\Gamma$. In [9], Gross shows that $A_f^{(\Gamma)}$ is K -isogenous to $\text{Res}_{H/K} A(p)$. Therefore, on the one hand, there is $\sigma \in \text{Gal}(H/K)$ such that $A(p)$ and ${}^\sigma C_\Gamma$ are \mathbb{Q}^{alg} -isomorphic. On the other hand, by Theorem 5.4, $A(p)$ and ${}^\sigma C_\Gamma$ are H -isogenous. Hence, $A(p)$ is H -isomorphic to ${}^\sigma C_\Gamma$ and the claim follows. \square

6 Canonical CM elliptic direction for $A(p)$

When the class number of K is greater than one, there are infinitely many elliptic directions in $S_2(\Gamma_0(p^2))$ attached to different parametrizations $J_0(p^2) \rightarrow A(p)$. Here,

we shall emphasize one of them (we call it canonical) in terms of a particular one-cocycle that can be constructed by means of the Dedekind eta-function.

Let \mathcal{O}_H be the ring of integers of the Hilbert class field H . For all $a \in K$ coprime with \mathfrak{p} , we denote by $\left(\frac{a}{\mathfrak{p}}\right)$ the Jacobi symbol $\left(\frac{m}{p}\right)$, where m is an integer such that $a \equiv m \pmod{\mathfrak{p}}$. One has $\eta(a) = \left(\frac{a}{\mathfrak{p}}\right)$. By [10], we know that there is a unique map $\delta : I(\mathfrak{p}) \rightarrow H$ with the following two requirements:

$$(i) \quad \delta(\mathfrak{a})^{12} = \Delta(\mathcal{O})/\Delta(\mathfrak{a}),$$

$$(ii) \quad \left(\frac{N_{H/K}(\delta(\mathfrak{a}))}{\mathfrak{p}}\right) = 1,$$

for all $\mathfrak{a} \in I(\mathfrak{p})$. Moreover, this map also satisfies the following conditions:

$$(iii) \quad \delta(\mathfrak{a})\mathcal{O}_H = \mathfrak{a}\mathcal{O}_H,$$

$$(iv) \quad \delta(\mathfrak{a} \cdot \mathfrak{b}) = \delta(\mathfrak{a}) \cdot \mathfrak{a}^{-1} \delta(\mathfrak{b}) \text{ for all } \mathfrak{a}, \mathfrak{b} \in I(\mathfrak{p}),$$

$$(v) \quad \delta(\overline{\mathfrak{a}}) = \overline{\delta(\mathfrak{a})} \text{ for all } \mathfrak{a} \in I(\mathfrak{p}).$$

By taking into account conditions (ii) and (iv), and since $[H : K]$ is odd, we also obtain:

$$(vi) \quad \text{for all } \mathfrak{a} \in P(\mathfrak{p}), \text{ one has } \delta(\overline{\mathfrak{a}}) \in K \text{ and } \left(\frac{\delta(\mathfrak{a})}{\mathfrak{p}}\right) = 1.$$

For every $\mathfrak{a} \in I(\mathfrak{p})$, we set

$$\lambda(\mathfrak{a}) := \mathfrak{a} \delta(\mathfrak{a}) = \frac{N(\mathfrak{a})}{\delta(\overline{\mathfrak{a}})}. \quad (5)$$

The map $\lambda : I(\mathfrak{p}) \rightarrow H$ also satisfies conditions (ii), (iii), (v), and (vi). But now conditions (i) and (iv) are replaced with

$$(i') \quad \lambda(\mathfrak{a})^{12} = N(\mathfrak{a})^{12} \frac{\Delta(\overline{\mathfrak{a}})}{\Delta(\mathcal{O}_K)},$$

and the one-cocycle condition:

$$(iv') \quad \lambda(\mathfrak{a} \cdot \mathfrak{b}) = \lambda(\mathfrak{a}) \cdot \mathfrak{a} \lambda(\mathfrak{b}), \text{ for all } \mathfrak{a}, \mathfrak{b} \in I(\mathfrak{p}).$$

Conditions (vi) and (iv') imply that the one-cocycle λ belongs to $[A_f]$ for all A_f of CM elliptic type and level p^2 .

Remark 6.1 Notice that the above one-cycle λ can be effectively computed by using the Dedekind eta-function on ideals (as Rodríguez-Villegas does in [13]), and it coincides with what Hajir denotes ϕ in Definition 2.3 in [11].

Let f denote the normalized newform in $S_2(\Gamma_0(p^2))$ attached to a Hecke character ψ whose eta-character has order 2. By Section 3, the splitting field L of A_f is H . Let $S_2(A_f)$ be the \mathbb{C} -vector space generated by the Galois conjugates of the newform f attached to ψ and let ω denote a Néron differential of Gross's elliptic curve $A(p)$.

Proposition 6.1 *Let f be as above. There is an optimal quotient $\pi : J_0(p^2) \longrightarrow A(p)$ such that $\pi^*(\omega) = c g(q) dq/q$, where*

$$g(q) = \sum_{(\mathfrak{a}, \mathfrak{p})=1} \delta(\mathfrak{a}) q^{N(\mathfrak{a})} \in S_2(A_f),$$

and $c \in \mathbb{Z}$ is a unit in $\mathbb{Z}[\frac{1}{2p}]$.

Proof. By Lemma 5.3, we have $[E : K] = [L : K]$ and, thus, all one-cocycles in $[A_f]$ are modular. Therefore, by Theorem 1.3 we have that

$$g(q) = \sum_{(\mathfrak{a}, \mathfrak{p})=1} \mathfrak{a}^{-1} \lambda(\mathfrak{a}) q^{N(\mathfrak{a})} = \sum_{(\mathfrak{a}, \mathfrak{p})=1} \delta(\mathfrak{a}) q^{N(\mathfrak{a})}$$

is a normalized cusp form in $S_2(A_f)$ for which there is an optimal elliptic quotient C_λ given by the lattice

$$\Lambda_g = \left\{ 2\pi i \int_\gamma g(z) dz : \gamma \in H_1(X_0(p^2), \mathbb{Z}) \right\}.$$

Since $\delta(\bar{\mathfrak{a}}) = \overline{\delta(\mathfrak{a})}$ for all $\mathfrak{a} \in I(\mathfrak{p})$, it follows that $g(q) \in H_0[[q]]$. Thus, $g(q) dq/q \in \Omega^1(X_0(p^2))_{/H_0}$. Hence, the natural morphism $\pi : X_0(p^2) \rightarrow C_\lambda$ is defined over H_0 . Notice that necessarily one has $\Lambda_g = \Omega \cdot \mathcal{O}_K$, for some $\Omega \in \mathbb{C}^*$. Indeed, \mathcal{O}_K is the only ideal \mathfrak{a} such that $j(\mathfrak{a}) = \overline{j(\mathfrak{a})}$ since $[H : K]$ is odd. Thus, we have $j(\Lambda_g) = j(\mathcal{O}_K)$. Since A_f is \mathbb{Q} -isogenous to $\text{Res}_{H_0/\mathbb{Q}}(C_\lambda)$ and to $\text{Res}_{H_0/\mathbb{Q}}(A(p))$, it follows that C_λ and $A(p)$ are H_0 -isogenous and, therefore, H_0 -isomorphic. Therefore, there exists $c \in H_0^*$ such that $\pi^*(\omega) = c g(q) dq/q$. It is clear that $\Delta(\Lambda_g) = -p^3 c^{12}$.

The Manin ideal attached to π is $c \mathcal{O}_{H_0}$ (we refer to Section 4 in [8] for more details on the Manin ideal). By Propositions 4.1 and 4.2 in [8], we know that $c \mathcal{O}_{H_0}$ is an integral ideal and it can only be divided by primes lying over 2 or p . Now, we want to prove that $c \in \mathbb{Z}$. Since $\pi^*(\omega/c) = g dq/q$, the one-cocycle attached to ω/c is λ . This means that for every $\mathfrak{a} \in I(\mathfrak{p})$ there is an isogeny of degree $N(\mathfrak{a})$,

$$\mu : \mathfrak{a}^{-1} C_\lambda \rightarrow C_\lambda,$$

such that $\mu^*(\omega/c) = \mathfrak{a}^{-1} \lambda(\mathfrak{a}) \cdot \mathfrak{a}^{-1}(\omega/c)$. Taking into account that $j(\mathfrak{a}) = \mathfrak{a}^{-1} j(\mathcal{O}_K)$, we obtain that the lattice corresponding to $\mathfrak{a}^{-1} C_\lambda$ is $\frac{1}{\delta(\mathfrak{a})} \cdot \Omega \mathfrak{a}$. Finally, we have that:

$$\mathfrak{a}^{-1} \Delta(\Omega \mathcal{O}_K) = \Delta\left(\frac{1}{\delta(\mathfrak{a})} \Omega \mathfrak{a}\right) = \delta(\mathfrak{a})^{12} \Delta(\Omega \mathfrak{a}) = \frac{\Delta(\mathcal{O}_K)}{\Delta(\mathfrak{a})} \Delta(\Omega \mathfrak{a}) = \Delta(\Omega \mathcal{O}_K).$$

Therefore, $\Delta(\Lambda) \in K \cap H_0 = \mathbb{Q}$ and $c^{12} \in \mathbb{Q}$. Since $\mathbb{Q}(c) \subseteq H$ is unramified outside p and there is not a real quadratic field of discriminant p , it follows that $c^3 \in \mathbb{Q}$. Finally, since H does not contain the 3rd roots of unity (recall $p > 3$), one obtains $c \in \mathbb{Q}$. \square

Remark 6.2 *Since $c \in K^*$, the one-cocycle attached to ω is also λ . In this sense, we say that the normalized cusp form g is the canonical cusp form attached to $A(p)$.*

For when the class number of K is 1 (that is, $p = 7, 11, 19, 43, 67, 163$), one has that π is defined over \mathbb{Q} and c coincides with the (classical) Manin constant. Then, $c = \pm 1$ in these cases since Manin's conjecture has been checked for all elliptic curves over \mathbb{Q} with conductor ≤ 130000 in Cremona's tables. We have computed c for the remaining primes $p \leq 100$ (that is, $p = 23, 31, 47, 59, 71, 79, 83$) and we have also obtained that $c = \pm 1$. It seems reasonable to expect $c = \pm 1$ for all $A(p)$.

Remark 6.3 *In general, as already mentioned, there are infinitely many normalized cusp forms $g' \in S_2(A_f)$ whose directions are pullbacks of $\Omega^1(A(p))$ under modular parametrizations $\pi' : A_f \rightarrow A(p)$. For each one of them, there is a one-cocycle λ' (cohomologous to λ) such that*

$$g' = \sum_{\mathfrak{a}} \mathfrak{a}^{-1} \lambda'(\mathfrak{a}) q^{N(\mathfrak{a})} \in S_2(A_f),$$

and a constant $c' \in H_0$ with $\pi'^*(\omega) = c' g'$. The concern on whether the constant c is ± 1 is already in [9], see Question 23.2.2 in pag. 81, but without fixing π' . However, $c' \neq \pm 1$ unless $\pi' = \pi$ (canonical) as in Theorem 6.1, although the Manin ideal attached to any $\pi' \neq \pi$ might still be \mathcal{O}_K as well.

We end this section giving an expression for the transcendental $\Omega \in \mathbb{C}^*$ attached to the lattice Λ of $A(p)$, which generalizes the one given by Gross in [9] for when K has class number one. Keeping the above notations, as in [11], we set

$$\rho := \prod_{\substack{\mathfrak{b} \in \text{Gal}(H/K) \\ (\mathfrak{b}, \mathfrak{p})=1}} \frac{\delta(\mathfrak{b})}{\psi(\mathfrak{b})}.$$

It is clear that ρ is well-defined, independent of the Galois conjugate of ψ , and $\rho \in \mathcal{O}_H^*$. Let h denote the class number of K , and consider

$$\{\mathcal{O}_K, \mathfrak{b}_1, \dots, \mathfrak{b}_{(h-1)/2}, \dots, \bar{\mathfrak{b}}_1, \dots, \bar{\mathfrak{b}}_{(h-1)/2}\}$$

a set of representatives of $\text{Gal}(H/K)$ with $(\mathfrak{b}_i, \mathfrak{p}) = 1$. Then, we can rewrite

$$\rho = \prod_{i=1}^{(h-1)/2} \frac{\delta(\mathfrak{b}_i) \delta(\bar{\mathfrak{b}}_i)}{N(\mathfrak{b}_i)} = \prod_{\mathfrak{b} \in \text{Gal}(H/K)} \frac{\delta(\mathfrak{b})}{\sqrt{N(\mathfrak{b})}}. \quad (6)$$

Indeed, since $\delta(\mathfrak{b})/\psi(\mathfrak{b})$ is independent of the class of \mathfrak{b} in $\text{Gal}(H/K)$, it suffices to prove that $\psi(\mathfrak{b}) \cdot \psi(\bar{\mathfrak{b}}) = N(\mathfrak{b})$. But this is a consequence of

$$\left(\frac{N(\mathfrak{b})}{p}\right) = \left(\frac{N(\mathfrak{b})}{p}\right)^h = \left(\frac{\beta}{\mathfrak{p}}\right) \left(\frac{\bar{\beta}}{\mathfrak{p}}\right) = \left(\frac{\beta}{\mathfrak{p}}\right)^2 = 1,$$

where $\beta \in K$ is a generator of \mathfrak{b}^h . Observe that ρ is a positive unit in $\mathcal{O}_{H_0}^*$.

Proposition 6.2 *Let $\Lambda = \Omega \cdot \mathcal{O}_K$ be the lattice attached to $A(p)$. Then,*

$$\Omega = \pm i^{(p+1)/4} \sqrt[h]{\rho \cdot (2\pi)^{(2h+1-p)/4} \cdot \sqrt{p}^{(1-3h)/2} \cdot \prod_{\substack{1 \leq m < p \\ \chi(m)=1}} \Gamma\left(\frac{m}{p}\right)},$$

where the h -th root is taken to be real.

Proof. By the Chowla-Selberg formula [2], we know that

$$\prod_{\mathfrak{a} \in \text{Gal}(H/K)} N(\mathfrak{a})^{-6} \Delta(\tau_{\mathfrak{a}}) = \left(\frac{2\pi}{p}\right)^{6h} \left(\prod_{m=1}^{p-1} \Gamma\left(\frac{m}{p}\right)^{\chi(m)}\right)^6,$$

where $\langle 1, \tau_{\mathfrak{a}} \rangle = \frac{1}{N(\mathfrak{a})} \mathfrak{a}$. Since λ is the one-cocycle attached to ω , we have that:

$$\Delta(\tau_{\mathfrak{a}}) = N(\mathfrak{a})^{12} \Delta(\mathfrak{a}) = N(\mathfrak{a})^{12} \Delta\left(\frac{\Omega}{\delta(\mathfrak{a})} \mathfrak{a}\right) \frac{\Omega^{12}}{\delta(\mathfrak{a})^{12}} = -p^3 \frac{N(\mathfrak{a})^{12}}{\delta(\mathfrak{a})^{12}} \Omega^{12}. \quad (7)$$

Combining (6), (7), and Gauss's identity

$$\prod_{i=1}^{n-1} \Gamma\left(\frac{i}{n}\right) = (2\pi)^{(n-1)/2} n^{-1/2},$$

the statement follows by taking into account that Ω lies in \mathbb{R} or $i\mathbb{R}$ according to $p \equiv -1 \pmod{8}$ or not (cf. [9]). \square

As by result, we obtain the following fact, which concludes the proof of Theorem 1.5.

Corollary 6.3 *With the above notations, one has*

$$\left\{ 2\pi i \int_{\gamma} g(z) dz : \gamma \in H_1(X_0(p^2), \mathbb{Z}) \right\} = \frac{1}{c} \cdot \Omega \cdot \mathcal{O}_K.$$

7 CM elliptic directions for non-trivial Nebentypus

In this section, we shall consider arbitrary Hecke characters mod \mathfrak{p} . Let ψ in \mathcal{X} and let η be its eta-character. Let f denote the normalized newform attached to ψ . In order to find the elliptic directions in $S_2(A_f)$, one needs to determine the modular one-cocycles λ_u in $[A_f]$. Then, the normalized cusp forms

$$g_u = \sum_{(\mathfrak{a}, \mathfrak{p})=1} \mathfrak{a}^{-1} \lambda_u(\mathfrak{a}) q^{N(\mathfrak{a})}$$

are the elliptic directions in $S_2(A_f)$. Recall that in the particular case $\eta^2 = 1$, all one-cocycles are modular. In general, as explained above, to find the modular one-cocycles amounts to an eigenvector problem. In our particular setting, the following lemma will be useful since it will allow to handle certain linear systems by means of a quotient polynomial ring.

Lemma 7.1 *Let M/F be a cyclic field extension of degree k . Fix τ a generator of $\text{Gal}(M/F)$, and let μ_k be the group of k th roots of unity. Let $\mathcal{E} = \text{End}_{F[\text{Gal}(M/F)]}(M)$ be the F -algebra of $\text{Gal}(M/F)$ -equivariant F -linear endomorphisms of M . One has:*

(i) *the map $\Theta : F[X]/(X^k - 1) \longrightarrow \mathcal{E}$ given by*

$$\Theta\left(\sum_{i=1}^k a_i X^i\right)(u) = \sum_{i=1}^k a_i \tau^i u, \quad \text{for all } u \in M,$$

is well-defined and an isomorphism of F -algebras.

(ii) *For every $p(X) \in F[X]/(X^k - 1)$, let $\mathcal{Z} = \{\zeta \in \mu_k : p(\zeta) = 0\}$. Then, the endomorphism $G = \Theta(p(X))$ diagonalizes and its characteristic polynomial is*

$$(-1)^k \prod_{i=1}^k (X - p(\zeta_k^i)),$$

where $\zeta_k = e^{2\pi i/k}$. We have $\dim_F \ker G = |\mathcal{Z}|$, and

$$\ker G = \Theta\left(\frac{X^k - 1}{\prod_{\zeta \in \mathcal{Z}} (X - \zeta)}\right)(M). \quad (8)$$

Proof. It is obvious that Θ is well-defined and a morphism of F -algebras. Chose $\alpha \in M$ such that $\{\tau^i \alpha\}_{1 \leq i \leq k}$ is a F -basis of M . The morphism Θ is injective because $\Theta(q(X)) = 0$ implies that $\Theta(q(X))(\alpha) = 0$ and, then, $q(X) = 0$. For a given $G \in \mathcal{E}$, we have that $G(\alpha) = \sum_{i=1}^k a_i \tau^i \alpha$ for some $a_i \in F$ and, thus, $G(u) = \sum_{i=1}^k a_i \tau^i u$ for all $u \in M$. Therefore, Θ is surjective and part (i) is proved.

We consider the F -algebra monomorphism $\Psi : \mathcal{E} \longrightarrow \text{End}_F F[X]/(X^k - 1)$ defined by $\Psi(G) = \widehat{G}$, where

$$\widehat{G}(q(X)) = \Theta^{-1}(G) \cdot q(X), \quad \text{for all } q(X) \in F[X]/(X^k - 1). \quad (9)$$

Now it suffices to prove part (ii) for \widehat{G} . Note that for any field extension F_0/F , the relation (9) allows us to consider \widehat{G} as a F_0 -linear endomorphism of $F_0[X]/(X^k - 1)$.

Let $G = \Theta(p(X))$. The set of eigenvalues of \widehat{G} is $\{p(\zeta_k^i) : 1 \leq i \leq k\}$. Indeed, if $\beta \in F_0$ is an eigenvalue of eigenvector $q(X) \in F_0[X]/(X^k - 1)$, then there exists $\zeta \in \mu_k$ such that $q(\zeta) \neq 0$ and, thus, $\beta = p(\zeta)$. Conversely, if $\beta = p(\zeta)$ for some $\zeta \in \mu_k$ then $q(X) = \prod_{\zeta' \in \mu_k \setminus \{\zeta\}} (X - \zeta')$ is an eigenvector with eigenvalue β . Notice that all eigenvalues of \widehat{G} are in $F_0 = F(\mu_k)$.

Now, let $\beta = p(\zeta)$ for some $\zeta \in \mu_k$ and we will prove that

$$\dim_{F_0} \ker(\widehat{G} - \beta \text{id}) = |\{\zeta \in \mu_k : p(\zeta) = \beta\}|,$$

which implies part (ii) except for the equality (8). Note that by a translation of \widehat{G} , we can (and do) assume $\beta = 0$. Then, one has

$$\begin{aligned} \ker \widehat{G} &= \{q(X) \in F_0[X]/(X^k - 1) : q(\zeta) = 0 \text{ for all } \zeta \in \mu_k \setminus \mathcal{Z}\} = \\ &= \{q(X) \in F_0[X]/(X^k - 1) : q(X) = \prod_{\zeta \in \mu_k \setminus \mathcal{Z}} (X - \zeta) r(X), \deg r < |\mathcal{Z}|\}. \end{aligned}$$

It follows that $\dim_{F_0} \ker \widehat{G} = |\mathcal{Z}|$ and $\ker \widehat{G} = \ker(\Psi \circ \Theta)(\prod_{\zeta \in \mathcal{Z}} (X - \zeta))$. Finally, the equality (8) is a consequence of the fact that $q(X) = \prod_{\zeta \in \mathcal{Z}} (X - \zeta) \in F[X]$ is coprime with $r(X) = (X^k - 1)/p(X)$ and $q(X) \cdot r(X)$ is zero in $F[X]/(X^k - 1)$. \square

Now, we focus our attention to the Hecke character $\psi \in \mathcal{X}$. For the sake of simplicity, let us assume that its eta-character satisfies $\text{ord}(\eta) = p - 1$. Since $\ker \eta$ is trivial, the corresponding field L is the ray class field of $K \bmod \mathfrak{p}$; that is, $L = H \cdot \mathbb{Q}(\zeta_p)$ (cf. Proposition 5.1). The cyclic group $\text{Gal}(L/H)$ has order $k := (p - 1)/2$. Also, let $\mathcal{E} = \text{End}_{H[\text{Gal}(L/H)]}(L)$ be the H -algebra of $\text{Gal}(L/H)$ -equivariant endomorphisms. After fixing a generator τ of $\text{Gal}(L/H)$, consider Θ as in Lemma 7.1. Finally, let $\lambda : I(\mathfrak{p}) \rightarrow L^*$ be the one-cocycle in Section 6. To find the elliptic directions in $S_2(A_f)$ turns out to be equivalent to find the twisted one-cocycles $\lambda_u(\mathfrak{a}) = \lambda(\mathfrak{a}) u / {}^{\mathfrak{a}}u$ which are modular. Note that now λ is not modular in $[A_f]$.

Proposition 7.2 *For all $u \in L^*$, the following conditions are equivalent:*

- (i) *the one-cocycle $\lambda_u(\mathfrak{a}) = \lambda(\mathfrak{a}) \frac{u}{{}^{\mathfrak{a}}u}$ is modular;*
- (ii) $u = \Theta\left(\frac{X^k - 1}{\Phi_k(X)}\right)(v)$, for some $v \notin \ker \Theta\left(\frac{X^k - 1}{\Phi_k(X)}\right)$.

In particular, for $u = \Theta\left(\frac{X^k - 1}{\Phi_k(X)}\right)(\zeta_p)$ the one-cocycle λ_u is modular. Here, $\Phi_k(X)$ denotes the k -th cyclotomic polynomial.

Proof. The values $u \in L^*$ for which λ_u is modular are the eigenvectors of the K -linear map

$$\text{pr}(u) = \sum_{\mathfrak{a} \in \text{Gal}(L/K)} \mathfrak{a}^{-1} \lambda(\mathfrak{a}) \left(\sum_{\sigma \in \Phi} \frac{1}{\sigma \psi(\mathfrak{a})} \right) \mathfrak{a}^{-1} u \quad (10)$$

with eigenvalue equal to $[L : K]$. Also, by Proposition 4.4, we know that $\text{pr}/[L : K]$ is a projector, pr diagonalizes, and its characteristic polynomial is

$$([L : K] - X)^{[E:K]} X^{[L:K] - [E:K]} = \left(([L : K] - X)^{\varphi(k)} X^{k - \varphi(k)} \right)^{[H:K]}.$$

By part (i) of Lemma 5.2, we can rewrite

$$\text{pr}(u) = \sum_{\mathfrak{a} \in \text{Gal}(L/H)} \mathfrak{a}^{-1} \lambda(\mathfrak{a}) \left(\sum_{\sigma \in \Phi} \frac{1}{\sigma \psi(\mathfrak{a})} \right) \mathfrak{a}^{-1} u.$$

Let $g \in \mathbb{Z}$ be a primitive root of $(\mathbb{Z}/p\mathbb{Z})^*$ such that $\eta(g) = \zeta$, where $\zeta = e^{\frac{\pi i}{k}}$. Since the set of principal ideals $\{\mathfrak{a}_j = g^{2j} \mathcal{O}_K : 1 \leq j \leq k\}$ is a set of representatives of $\text{Gal}(L/H)$ and $\lambda(g^{2j} \mathcal{O}_K) = g^{2j}$, we have

$$G(u) := \frac{\text{pr}(u)}{[H : K]} = \frac{1}{[H : K]} \sum_{j=1}^k \left(\sum_{\sigma \in \Phi} \sigma \zeta^{-2j} \right) \mathfrak{a}_j^{-1} u = \sum_{j=1}^k \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\zeta^{-2j}) \mathfrak{a}_j^{-1} u.$$

Hence, G belongs to \mathcal{E} and its characteristic polynomial has roots 0 and k with multiplicities $k - \varphi(k)$ and $\varphi(k)$, respectively.

Now, we fix the generator $\tau = g^{-2} \mathcal{O}_K$ of $\text{Gal}(L/H)$ and apply Lemma 7.1 to the endomorphism $G - k\text{Id} \in \mathcal{E}$. It follows that the set

$$\mathcal{Z} = \{\zeta' \in \mu_k : \sum_{j=1}^k \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\zeta^{-2j}) (\zeta')^{2j} - k = 0\}$$

has cardinality $|\mathcal{Z}| = \varphi(k)$. Letting $\zeta_k = \zeta^2$, we claim that

$$\mathcal{Z} = \{\zeta_k^j : 1 \leq j < k, \gcd(j, k) = 1\}.$$

Since $\text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ acts transitively on \mathcal{Z} and $|\mathcal{Z}| = \varphi(k)$, it suffices to prove that $\zeta_k \in \mathcal{Z}$. Indeed, one checks:

$$\sum_{j=1}^k \left(\sum_{i \in (\mathbb{Z}/k\mathbb{Z})^*} \zeta_k^{-ji} \right) \zeta_k^j = \sum_{j=1}^k \left(\sum_{i \in (\mathbb{Z}/k\mathbb{Z})^*} \zeta_k^{(1-i)j} \right) = \sum_{j=1}^k \left(\sum_{i \in (\mathbb{Z}/k\mathbb{Z})^*} \zeta_k^{ij} \right) = k.$$

Then, from Lemma 7.1, we obtain

$$\{u \in L: \text{pr}(u) = [L:K]u\} = \{u = \Theta\left(\frac{X^k-1}{\Phi_k(X)}\right)(v): v \in L\}.$$

Note that the image of $\Theta\left(\frac{X^k-1}{\Phi_k(X)}\right)$ is independent of the choice of the generator τ in $\text{Gal}(L/H)$. It can be easily checked that $\Theta((X^k-1)/\Phi_k(X))$ vanishes on H , which implies that $\Theta((X^k-1)/\Phi_k(X))(\zeta_p)$ is non-zero since the class of the polynomial $(X^k-1)/\Phi_k(X)$ in $L[X]/(X^k-1)$ is non-zero. \square

Example: take $p = 7$, so that $K = \mathbb{Q}(\sqrt{-7})$ has class number one. Let ψ in \mathcal{X} with eta-character satisfying $\eta(3) = e^{2\pi i/6}$. Its corresponding newform $f = \sum \psi((a))q^{N(a)} \in S_2(\Gamma_1(49))$ has Nebentypus ε of order 3; note that $\psi((a)) = a\eta(a)$ for all $a \in \mathcal{O}_K$. The one-cocycle λ satisfies $\lambda((a)) = a$ with the unique choice of sign for a such that the symbol $(a/\sqrt{-7}) = 1$. This one-cocycle is not modular for ψ (in fact, it is modular for the Hecke character in \mathcal{X} with eta-character of order 2 in which case the (unique) elliptic direction coincides with the rational newform in $S_2(\Gamma_0(49))$ giving rise to the elliptic curve 49A1 in Cremona's notation.) Thus, we need to twist λ by a coboundary in order to get a modular one-cocycle. According to Proposition 7.2, we can take, for instance, $u = \Theta(X-1)(\zeta_7) = \zeta_7^2 - \zeta_7$ and the cuspidal form $g_u = \sum a^{-1} \lambda_u(a) q^{N(a)} = \sum \lambda((a))^{(a^2)} u / u q^{N(a)} \in S_2(\Gamma_1(49))$ is an elliptic direction of A_f . A computer calculation shows the lattice Λ for the corresponding elliptic optimal quotient from $\text{Jac}(X_{\Gamma_\varepsilon})$ satisfies: $c_4(\Lambda) = c_4(A(7))u^4$, and $c_6(\Lambda) = c_6(A(7))u^6$.

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